

The impact of the diagonals of polynomial forms on limit theorems with long memory

Shuyang Bai Murad S. Taqqu

April 1, 2014

Abstract

We start with an i.i.d. sequence and consider the product of two polynomial-forms moving averages based on that sequence. The coefficients of the polynomial forms are asymptotically slowly decaying homogeneous functions so that these processes have long memory. The product of these two polynomial forms is a stationary nonlinear process. We want to obtain limit theorems for the normalized sums of this nonlinear process in two cases: exclusion of the diagonal terms of the polynomial form, or inclusion. In either case, if the product has long memory, then the limits are given by a Wiener chaos, but these limits are quite different. If the diagonals are excluded, then the limit is expressed as in the product formula of two Wiener-Itô integrals. When the diagonals are included, the limit stochastic integrals are typically due to a single factor of the product, namely the one with the strongest memory.

1 Introduction

Let $X(n)$ be a stationary process with mean 0 and finite variance. We are interested in the following weak convergence of normalized partial sum to a process $Z(t)$:

$$\frac{1}{A(N)} \sum_{n=1}^{[Nt]} X(n) \Rightarrow Z(t) \quad (1)$$

as $N \rightarrow \infty$ where $A(N) \rightarrow \infty$ is a suitable normalization. The limit $Z(t), t \geq 0$ if it exists, has stationary increments and is self-similar with some index $H > 0$, that is, for any $a > 0$, $\{Z(at), t \geq 0\}$ and $\{a^H Z(t), t \geq 0\}$ have the same finite-dimensional distributions. The parameter H is called the *memory parameter*¹ of the process $X(n)$ and the *Hurst index* or *self-similarity parameter* of the limit process $Z(t)$.

When the dependence in $X(n)$ is weak, one typically ends up in (1) with $A(N) = \left(\text{Var}[\sum_{n=1}^N X(n)] \right)^{1/2} \sim cN^{1/2}$ as $N \rightarrow \infty$ for some $c > 0$, and $Z(t)$ is the Brownian motion. These types of limit theorems are often called *central limit theorems*.

When, however, the dependence in $X(n)$ is so strong that $\text{Var}[\sum_{n=1}^N X(n)]$ grows faster than the linear speed N , and typically as N^{2H} with $H \in (1/2, 1)$, the limit process $Z(t)$ in (1) is no longer Brownian motion. $Z(t)$ is in this case a self-similar process with stationary increments which has a *Hurst index* H (see Embrechts and Maejima [5]). This type of limit theorems involving non-Brownian limits are often called *non-central limit theorems*. When the process $X(n)$ is nonlinear and has long memory, the limit $Z(t)$ can be non-Gaussian (e.g., Dobrushin and Major [4], Taqqu [10], Surgailis [9]).

Key words Long memory; Self-similar processes; Volterra; Wiener; Non-central limit theorem; Diagonals

2010 AMS Classification: 60G18, 60F05

¹A precise definition of memory parameter is given in Definition 4.3.

In Bai and Taqqu [1], a non-central limit theorem is established for an off-diagonal polynomial-form process called *k*-th order *discrete chaos process*:

$$Y'(n) = \sum_{0 < i_1, \dots, i_k < \infty}^{\prime} a(i_1, \dots, i_k) \epsilon_{n-i_1} \dots \epsilon_{n-i_k}, \quad (2)$$

where the prime \prime indicates that we do not sum on the diagonals $i_p = i_q$, $p \neq q$, the noise ϵ_i 's are i.i.d. random variables with mean 0 and variance 1, and $a(\cdot)$ is asymptotically some homogeneous function g called *generalized Hermite kernel* (GHK). The limit $Z(t)$, called a *generalized Hermite process*, is expressed by a *k*-tuple *Wiener-Itô integral*:

$$Z(t) = \int_{\mathbb{R}^k}^{\prime} \int_0^t g(s - x_1, \dots, s - x_k) 1_{\{s > x_1, \dots, s > x_k\}} ds B(dx_1) \dots B(dx_k), \quad (3)$$

where the prime \prime indicates that we do not integrate on the diagonals $x_p = x_q$, $p \neq q$, and $B(\cdot)$ is Brownian random measure. These processes $Z(t)$ include the *Hermite process* considered in Dobrushin and Major [4], Taqqu [10] and Surgailis [9].

In Bai and Taqqu [2], a non-central limit theorem is established for a polynomial-form process called *k*-th order *discrete Volterra process*:

$$Y(n) = \sum_{0 < i_1, \dots, i_k < \infty} a(i_1, \dots, i_k) \epsilon_{n-i_1} \dots \epsilon_{n-i_k}, \quad (4)$$

which differs from $Y'(n)$ in (2) by including the diagonals, and where $a(\cdot)$ is asymptotically $g(\cdot)$, some special type of generalized Hermite kernel called *generalized Hermite kernel of Class (B)* (GHK(B)). The limit $Z(t)$ can be heuristically thought as (3) with diagonals included, and is precisely expressed as a *k*-tuple *centered Wiener-Stratonovich integral*, which is a linear combination of certain Wiener-Itô integrals of orders lower than or equal to k (see Bai and Taqqu [2]).

In this paper we contrast the effect of two types of stationary sequences in the limit theorem (1). The first stationary sequence is:

$$X(n) = Y'_1(n)Y'_2(n), \quad (5)$$

that is, a product of two long memory chaos processes (2) which *exclude* the diagonals. The second stationary sequence is:

$$X(n) = Y_1(n)Y_2(n), \quad (6)$$

that is, a product of two long memory processes in (4) which *include* the diagonals. We show that in the case (5), the limit in (1) is expressed as Wiener-Itô integrals which can be obtained by using a rule similar to that used for computing the product of two Wiener-Itô integrals. In fact, if the stationary sequences $Y'_1(n)$ and $Y'_2(n)$ have respectively memory parameters $H_1, H_2 \in (1/2, 1)$ with $H_1 + H_2 > 3/2$, then the limit in (1) has Hurst index $H = H_1 + H_2 - 1 \in (1/2, 1)$. In the case (6), in contrast, the limit stochastic integrals are typically due to a single factor $Y_1(n)$ or $Y_2(n)$, namely, the one with the strongest memory parameter. The Hurst index of the limit is then $\max(H_1, H_2) \in (1/2, 1)$ which is always greater than $H_1 + H_2 - 1$.

The paper is organized as follows. Section 2 contains some background. We state the main results in Section 3, namely, Theorem 3.4 for processes without diagonals, and Theorem 3.5 for processes with diagonals. Section 4 provides some preliminary results used in the proofs. Section 5 contains the proof of Theorem 3.4, and Section 6 contains the proof of Theorem 3.5.

2 Background

The following notation will be used throughout: $\mathbf{0}$ denotes the zero vector $(0, 0, \dots, 0)$ and $\mathbf{1} = (1, 1, \dots, 1)$ denotes the vector with ones in every component. For two vectors \mathbf{x} and \mathbf{y} in the same dimension, we write $\mathbf{x} \leq \mathbf{y}$ (or $<$, \geq , $>$) if the inequality holds componentwise. We let $[x] = \sup\{n \in \mathbb{Z} : n \leq x\}$ for any real x and a real vector $\mathbf{x} = (x_1, \dots, x_k)$, we define $[\mathbf{x}] = ([x_1], \dots, [x_k])$. The notation 1_A denotes the indicator function of a set A . The value of a constant $C > 0$ or $c > 0$ may change from line to line.

In Bai and Taqqu [1], the following classes of functions were introduced:

Definition 2.1. A measurable function g defined on \mathbb{R}_+^k is called a *generalized Hermite kernel* (GHK) with homogeneity exponent

$$\alpha \in \left(-\frac{k+1}{2}, -\frac{k}{2}\right), \quad (7)$$

if it satisfies

1. $g(\lambda \mathbf{x}) = \lambda^\alpha g(\mathbf{x})$, $\forall \lambda > 0$;
2. $\int_{\mathbb{R}_+^k} |g(\mathbf{1} + \mathbf{x})g(\mathbf{x})| d\mathbf{x} < \infty$;

A GHK g is said to belong to Class (B) (abbreviated as GHK(B)), if g is a.e. continuous on \mathbb{R}_+^k and $|g(\mathbf{x})| \leq c \|\mathbf{x}\|^\alpha = c(x_1 + \dots + x_k)^\alpha$ ($\|\cdot\|$ is the L^1 -norm) for some constant $c > 0$.

Remark 2.2. As was shown in Theorem 3.5 of Bai and Taqqu [1], if g is a GHK, then

$$\int_0^t |g(s\mathbf{1} - \mathbf{x})| 1_{\{s\mathbf{1} > \mathbf{x}\}} ds < \infty$$

for a.e. $\mathbf{x} \in \mathbb{R}^k$, and the function

$$h_t(\mathbf{x}) := \int_0^t g(s\mathbf{1} - \mathbf{x}) 1_{\{s\mathbf{1} > \mathbf{x}\}} ds \in L^2(\mathbb{R}^k).$$

Using a GHK, one can define a self-similar process with stationary increments on a Wiener chaos as follows:

Definition 2.3. Let g be a GHK on \mathbb{R}_+^k with homogeneity exponent $\alpha \in (-\frac{k+1}{2}, -\frac{k}{2})$, then (3) is called a *generalized Hermite process* $Z(t)$. It is self-similar with Hurst index

$$H = \alpha + k/2 + 1. \quad (8)$$

Example 2.4. If $g(\mathbf{x}) = \prod_{j=1}^k x_j^\gamma$, where $-1/2 - 1/k < \gamma < -1/2$, then $Z(t)$ in (3) is the *Hermite process* considered in Dobrushin and Major [4] and Taqqu [10].

Note that GHK(B) does not include the kernel in Example 2.4. We use a GHK(B) because of its boundedness property. The subclass of GHK(B) is, in fact, a dense subset in the whole class of GHK (see Remark 3.17 of Bai and Taqqu [1]).

We now state two limit theorems, the first for the discrete chaos process $Y'(n)$ defined in (2) where the diagonals are excluded, and the second for the Volterra process $Y(n)$ defined in (4) which includes the diagonals.

Suppose that g is a GHK(B) on \mathbb{R}_+^k , $L(\cdot)$ is a bounded function defined on \mathbb{Z}_+^k such that $\lim_{n \rightarrow \infty} L([n\mathbf{x}] + \mathbf{B}(n)) = 1$ for any $\mathbf{x} \in \mathbb{R}_+^k$ and any \mathbb{Z}_+^k -valued bounded function $\mathbf{B}(n)$, and suppose that the coefficient $a(\cdot)$ in (2) is given by

$$a(\mathbf{i}) = g(\mathbf{i})L(\mathbf{i}). \quad (9)$$

Proposition 2.5 (Theorem 6.5 of Bai and Taqqu [1]). *One has then the following weak convergence in $D[0, 1]$:*

$$\frac{1}{N^H} \sum_{n=1}^{[Nt]} Y'(n) \Rightarrow Z(t) := I_k(h_t), \quad (10)$$

where $H = \alpha + k/2 + 1 \in (1/2, 1)$,

$$h_t(\mathbf{x}) = \int_0^t g(s\mathbf{1} - \mathbf{x}) 1_{\{s\mathbf{1} > \mathbf{x}\}} ds \quad (11)$$

with g as in (9), and $I_k(\cdot)$ denotes the k -tuple Wiener-Itô integral, so that $Z(t)$ is a generalized Hermite process (3).

We now consider the limit when the diagonals are included. If g is GHK(B) on \mathbb{R}_+^k and is in addition symmetric, we define the following function g_r by identifying r pairs of variables of g and integrating them out, as follows:

$$g_r(\mathbf{x}) = \int_{\mathbb{R}_+^r} g(y_1, y_1, \dots, y_r, y_r, x_1, \dots, x_{k-2r}) d\mathbf{y}. \quad (12)$$

In Bai and Taqqu [2], a non-central limit theorem was established for the Volterra process $Y(n)$ in (4). Let $a(\cdot) = g(\cdot)L(\cdot)$ in (4) be given as in (9) assuming in addition that g is symmetric.

Proposition 2.6 (Theorem 6.2 of Bai and Taqqu [2]). *One has the following weak convergence in $D[0, 1]$:*

$$\frac{1}{N^H} \sum_{n=1}^{[Nt]} Y(n) \Rightarrow Z(t) := \sum_{0 \leq r < k/2} d_{k,r} Z_{k-2r}(t) \quad (13)$$

where $H = \alpha + k/2 + 1 \in (1/2, 1)$,

$$d_{k,r} = \frac{k!}{2^r (k-2r)! r!}, \quad (14)$$

and

$$Z_{k-2r}(t) := \int_{\mathbb{R}^{k-2r}} g_r(s\mathbf{1} - \mathbf{x}) 1_{\{s\mathbf{1} > \mathbf{x}\}} ds B(dx_1) \dots B(dx_k) \quad (15)$$

is a $(k-2r)$ -th order generalized Hermite process defined by g_r in (12).

Remark 2.7. The limit process $Z(t)$ in (13) can be simply expressed in terms of a centered Wiener-Stratonovich integral $\mathring{I}_k^c(\cdot)$ as

$$Z(t) = \mathring{I}_k^c(h_t), \quad (16)$$

where h_t is as in (11), and where $\mathring{I}_k^c(\cdot) = \sum_{0 \leq r < k/2} d_{k,r} \mathring{I}_{k-2r}(\tau^r \cdot)$. The integral $\mathring{I}_k^c(\cdot)$ differs from the Wiener-Stratonovich integral $\mathring{I}_k(\cdot) := \sum_{r=0}^{[k/2]} d_{k,r} \mathring{I}_{k-2r}(\tau^r \cdot)$ introduced in Hu and Meyer [7] by excluding the term $r = k/2$ when k is even. Here the operator τ^r identifies r pairs of variables of h and integrates them out (see Bai and Taqqu [2]). The operator τ^r is often called a “trace operator”.

3 Statement of the main results

We state here the main results, and defer the proofs to Sections 5 and 6. In the statement of the results, the following expressions are used:

Definition 3.1. Let $X(n)$ be a stationary process with finite variance. We say that:

1. $X(n)$ satisfies a *central limit theorem* (CLT), if

$$N^{-1/2} \sum_{n=1}^{[Nt]} [X(n) - \mathbb{E}X(n)] \Rightarrow \sigma B(t) \quad (17)$$

in $D[0, 1]$, where $\sigma^2 = \sum_{n=-\infty}^{\infty} \text{Cov}(X(n), X(0))$;

2. $X(n)$ satisfies a *non-central limit theorem* (NCLT) with a Hurst index $H \in (1/2, 1)$ and limit $Z(t)$, if

$$N^{-H} \sum_{n=1}^{[Nt]} [X(n) - \mathbb{E}X(n)] \Rightarrow Z(t) \quad (18)$$

in $D[0, 1]$.

Remark 3.2. In Case 1 above, the “long-run variance” σ^2 can be 0. In this case, we understand the limit theorem as degenerate (the normalization $N^{-1/2}$ is too strong). We do not consider here limit theorems involving a Hurst index $H < 1/2$. In Case 2, the limit in (18) may be fractional Brownian motion.

We now consider separately the cases where the diagonals of the polynomial forms are excluded (chaos processes) and when they are included (Volterra processes).

3.1 Limit theorem for a product of long-memory chaos processes

Suppose that we have the following two discrete chaos processes (off-diagonal polynomial forms):

$$Y_1'(n) = \sum_{\mathbf{i} \in \mathbb{Z}_+^{k_1}} a^{(1)}(\mathbf{i}) \epsilon_{n-i_1} \dots \epsilon_{n-i_{k_1}}, \quad Y_2'(n) = \sum_{\mathbf{i} \in \mathbb{Z}_+^{k_2}} a^{(2)}(\mathbf{i}) \epsilon_{n-i_1} \dots \epsilon_{n-i_{k_2}},$$

where we assume that $a^{(j)} = g^{(j)}L^{(j)}$ as in (9) is symmetric, where $g^{(j)}$ is a symmetric GHK(B) with homogeneity exponent $\alpha_j \in (-k_j/2 - 1/2, -k_j/2)$, $j = 1, 2$. Definition 2.3 suggests the following terminology:

Definition 3.3. The index

$$H = \alpha + k/2 + 1 \in (1/2, 1) \quad (19)$$

is called the *associated Hurst index* of the coefficient $a(\cdot) = g(\cdot)L(\cdot)$ in (9).

The associated Hurst indices of the coefficients in $Y_1'(n)$ and $Y_2'(n)$ will determine the Hurst index of the limit process $Z(t)$ in (1).

We want to obtain a limit theorem for the normalized partial sum of the product process:

$$X(n) := Y_1'(n)Y_2'(n). \quad (20)$$

Theorem 3.4. Let $X(n)$ be the product process in (20). Suppose that H_j is the associated Hurst index of $a^{(j)}(\cdot)$, $j = 1, 2$, and assume that $\mathbb{E}|\epsilon_i|^{4+\delta} < \infty$ for some $\delta > 0$.

1. If $H_1 + H_2 < 3/2$, then $X(n)$ satisfies the CLT (17);
2. If $H_1 + H_2 > 3/2$, then $X(n)$ satisfies the NCLT (18) with Hurst index $H = H_1 + H_2 - 1$ and limit

$$Z(t) = \sum_{r=0}^k r! \binom{k_1}{r} \binom{k_2}{r} I_{k_1+k_2-2r}(h_{t,r}), \quad (21)$$

where $k = k_1 \wedge k_2$ if $k_1 \neq k_2$, and $k = k_1 - 1$ if $k_1 = k_2$. The integrand $h_{t,r}$ above is defined as:

$$h_{t,r}(\mathbf{x}) = \int_0^t \left(g^{(1)} \otimes_r g^{(2)} \right) (s\mathbf{1} - \mathbf{x}) 1_{\{s\mathbf{1} > \mathbf{x}\}} ds, \quad (22)$$

where

$$g^{(1)} \otimes_r g^{(2)}(\mathbf{x}) := \int_{\mathbb{R}_+^r} g^{(1)}(y_1, \dots, y_r, x_1, \dots, x_{k_1-r}) g^{(2)}(y_1, \dots, y_r, x_{k_1-r+1}, \dots, x_{k_2+k_1-2r}) d\mathbf{y} \quad (23)$$

is a GHK, and when $r = 0$, (23) is understood as the tensor product $g^{(1)} \otimes g^{(2)}$. When $r > 0$ in (23), we identify r variables of $g^{(1)}$ and $g^{(2)}$ and integrate over them.

This theorem is proved in Section 5.

3.2 Limit theorem for a product of long-memory Volterra processes

Let now

$$X(n) = Y_1(n)Y_2(n) \quad (24)$$

where

$$Y_1(n) = \sum_{\mathbf{i} \in \mathbb{Z}_+^{k_1}} a^{(1)}(\mathbf{i}) \epsilon_{n-i_1} \dots \epsilon_{n-i_{k_1}}, \quad Y_2(n) = \sum_{\mathbf{i} \in \mathbb{Z}_+^{k_2}} a^{(2)}(\mathbf{i}) \epsilon_{n-i_1} \dots \epsilon_{n-i_{k_2}}.$$

We assume that $a^{(j)} = g^{(j)}L^{(j)}$ in (9) is symmetric, and $g^{(j)}$ is a symmetric GHK(B) with homogeneity exponent $\alpha_j \in (-k_j/2 - 1/2, -k_j/2)$, $j = 1, 2$. In this case, we can write

$$X(n) = \sum_{\mathbf{i} \in \mathbb{Z}_+^k} a(\mathbf{i}) \epsilon_{i_1} \dots \epsilon_{i_k},$$

where $k = k_1 + k_2$, and

$$a = a^{(1)} \otimes a^{(2)}. \quad (25)$$

Let \mathcal{C}_1^2 to be the collection of partitions of the set $\{1, \dots, k_1\}$ such that each set in the partition contains at least 2 elements, and similarly let \mathcal{C}_2^2 be the same thing for $\{k_1 + 1, \dots, k_1 + k_2\}$. Any partition $\pi \in \mathcal{C}_j^2$ can be expressed as $\pi = (P_1, \dots, P_m)$, where P_i , $i = 1, \dots, m$, are subsets ordered according to their smallest elements. For example, if $\pi = \{\{1, 4\}, \{2, 3\}\}$, then $P_1 = \{1, 4\}$ and $P_2 = \{2, 3\}$. Let

$$c_j = \sum_{\pi \in \mathcal{C}_j^2} \sum_{\mathbf{i} > \mathbf{0}} a_{\pi}^{(j)}(\mathbf{i}) \mu_{\pi}, \quad j = 1, 2, \quad (26)$$

where $\mu_{\pi} = \mu_{p_1} \dots \mu_{p_m}$ with $\mu_p = \mathbb{E} \epsilon_i^p$ and $p_i = |P_i| \geq 2$ if $\pi = (P_1, \dots, P_m)$, and where $a_{\pi}^{(j)}(\cdot)$ denotes $a^{(j)}$ with its variables identified according to the partition π (see (50) below).

The limit theorem for the normalized partial sum of the centered $X(n)$ in (24) includes several cases. We shall use the centered multiple Wiener-Stratonovich integral $\mathring{I}_k^c(\cdot)$ introduced in (16). The theorem states that except for some low-dimensional cases (Cases 1-4), the limit is up to some constant the same as the limit for a single factor, one with the highest H_j (Cases 5-7).

Theorem 3.5. *Let $X(n)$ be the product process in (24), where $a^{(j)}$ has associated Hurst index $H_j = \alpha_j + k_j/2 + 1 \in (1/2, 1)$ (Definition 3.3). Assume $\mathbb{E}|\epsilon_i|^{k_1+k_2+\delta} < \infty$ for some $\delta > 0$. Then using the language of Definition 3.1,*

1. *if $k_1 = 1, k_2 = 1$, and $H_1 + H_2 < 3/2$, then $X(n)$ satisfies a CLT (17);*
2. *if $k_1 = 1, k_2 = 1$, and $H_1 + H_2 > 3/2$, then $X(n)$ satisfies a NCLT (18) with Hurst index $H_1 + H_2 - 1$ and limit $Z(t) = \int_{\mathbb{R}^2}' \int_0^t g_1(s - x_1) g_2(s - x_2) 1_{\{s \mathbf{1} > \mathbf{x}\}} ds B(dx_1) B(dx_2)$ (non-symmetric Rosenblatt process);*
3. *if $k_1 \geq 2, k_2 = 1$, and if c_1 in (26) is nonzero, then $X(n)$ satisfies a NCLT (18) with Hurst index H_2 and limit $Z(t) = c_1 \int_{\mathbb{R}} \int_0^t g_2(s - x) 1_{\{s > x\}} ds B(dx)$ (fractional Brownian motion);*
4. *if $k_1 = 1, k_2 \geq 2$, and if c_2 in (26) is nonzero, then $X(n)$ satisfies a NCLT (18) with Hurst index H_1 and limit $Z(t) = c_2 \int_{\mathbb{R}} \int_0^t g_1(s - x) 1_{\{s > x\}} ds B(dx)$ (fractional Brownian motion);*
5. *if $k_1 \geq 2, k_2 \geq 2, H_1 > H_2$, and if c_2 in (26) is nonzero, then $X(n)$ satisfies a NCLT (18) with Hurst index H_1 , and the limit $Z(t) = c_2 \mathring{I}_{k_1}^c(h_{t,1})$, where $h_{t,1}(\mathbf{x}) = \int_0^t g_1(s \mathbf{1} - \mathbf{x}) 1_{\{s \mathbf{1} > \mathbf{x}\}} ds$;*
6. *if $k_1 \geq 2, k_2 \geq 2, H_1 < H_2$, and if c_1 in (26) is nonzero, then $X(n)$ satisfies a NCLT (18) with Hurst index H_2 , and the limit $Z(t) = c_1 \mathring{I}_{k_2}^c(h_{t,2})$, where $h_{t,2}(\mathbf{x}) = \int_0^t g_2(s \mathbf{1} - \mathbf{x}) 1_{\{s \mathbf{1} > \mathbf{x}\}} ds$.*
7. *if $k_1 \geq 2, k_2 \geq 2, H_1 = H_2$, and if at least one of the c_j 's in (26) is nonzero, then $X(n)$ satisfies a NCLT (18) with Hurst index $H_1 = H_2$, and the limit $Z(t) = c_1 \mathring{I}_{k_2}^c(h_{t,2}) + c_2 \mathring{I}_{k_1}^c(h_{t,1})$.*

Remark 3.6. These constants c_j 's in the theorem are nonzero if, for example, every $a^{(j)}(\mathbf{i}) > 0$, $j = 1, 2$.

The theorem, which is proved in Section 6, seems bewildering at first glance. But there is structure into it. The cases 3 and 4 are symmetric, and so are the cases 5 and 6. Case 1 involves short-range dependence, while all the other cases involve long-range dependence. Case 2 involves the non-symmetric Rosenblatt process, originally introduced by Maejima and Tudor [8]. Cases 3 and 4 involve fractional Brownian motion since one of the orders k equals 1. The typical cases are 5 (and 6). In these cases, quite surprisingly, it is not the orders k_1 or k_2 that matter, but the process $Y_1(n)$ or $Y_2(n)$ in (24) with the highest H . In the boundary case 7, where $H_1 = H_2$, they both contribute.

4 Preliminary results

A central idea in establishing the limit theorems is to involve the *non-symmetric discrete chaos processes* which generalizes the chaos process in (2) by allowing different sequences of noises. We shall now define it. Let $\epsilon_i = (\epsilon_i^{(1)}, \dots, \epsilon_i^{(k)})$ be an i.i.d. vector where each component has mean 0 and finite variance. The components $\epsilon_i^{(1)}, \dots, \epsilon_i^{(k)}$ are typically dependent. Introduce the following non-symmetric discrete chaos process

$$Y'(n) = \sum_{0 < i_1, \dots, i_k < \infty}^l a(i_1, \dots, i_k) \epsilon_{n-i_1}^{(1)} \dots \epsilon_{n-i_k}^{(k)}, \quad (27)$$

where $\sum_{i \in \mathbb{Z}_+^k} a(i)^2 < \infty$ so that $X'(n)$ is well-defined in the $L^2(\Omega)$ -sense. Let $\Sigma(i, j) = \mathbb{E} \epsilon_n^{(i)} \epsilon_n^{(j)}$. The autocovariance of $Y'(n)$ is then given by

$$\gamma(n) = \sum_{\sigma} \sum_{0 < i_1, \dots, i_k < \infty}^l a(i_1, \dots, i_k) a(i_{\sigma(1)} + n, \dots, i_{\sigma(k)} + n) \Sigma(i_1, i_{\sigma(1)}) \dots \Sigma(i_k, i_{\sigma(k)}), \quad (28)$$

where in the summation σ runs over all the $k!$ permutations of $\{1, \dots, k\}$. The following lemma is useful for studying the asymptotic properties of the covariance of $X'(n)$:

Lemma 4.1. *Suppose that in (27), there exist constant $c_0 > 0$ and $\gamma_j < -1/2$, $j = 1, \dots, k$, such that*

$$|a(i_1, \dots, i_k)| \leq c_0 i_1^{\gamma_1} \dots i_k^{\gamma_k}. \quad (29)$$

Let

$$H^* = \alpha + k/2 + 1 \text{ with } \alpha = \sum_{j=1}^k \gamma_j. \quad (30)$$

- If $H^* < 1/2$, then $\sum_{n=-\infty}^{\infty} |\gamma(n)| < \infty$, and $\text{Var}[\sum_{n=1}^N Y'(n)] \leq c_1 N$ for some $c_1 > 0$;
- If $H^* > 1/2$, then $|\gamma(n)| \leq c_2 n^{2H^*-2}$ for some $c_2 > 0$, and $\text{Var}[\sum_{n=1}^N Y'(n)] \leq c_3 N^{2H^*}$ for some $c_3 > 0$.

Proof. The case $H^* < 1/2$ was proved in Proposition 5.4 in Bai and Taqqu [2].

In the case $H^* > 1/2$, let $\widetilde{|a|}$ be the symmetrization of $|a|(\mathbf{i}) := |a(\mathbf{i})|$, then for $n \geq 0$, by (28) and (29),

$$\begin{aligned} |\gamma(n)| &\leq C_0 \sum_{\mathbf{i} \in \mathbb{Z}_+^k} \widetilde{|a|}(\mathbf{i} + n\mathbf{1}) \widetilde{|a|}(\mathbf{i}) \leq C_1 \sum_{\sigma} \sum_{i_1=1}^{\infty} \dots \sum_{i_k=1}^{\infty} (i_1 + n)^{\gamma_1} \dots (i_k + n)^{\gamma_k} i_1^{\gamma_{\sigma(1)}} \dots i_k^{\gamma_{\sigma(k)}} \\ &\leq C_2 \sum_{\sigma} n^{\gamma_1 + \gamma_{\sigma(1)} + 1} \dots n^{\gamma_k + \gamma_{\sigma(k)} + 1} = C_3 n^{2\alpha + k} = C_3 n^{2H^* - 2}, \end{aligned}$$

where C_i 's are positive constants, and σ in the summation runs over all the permutations of $\{1, \dots, k\}$. $\text{Var}[\sum_{n=1}^N Y'(n)] \leq c_3 N^{2H^*}$ then follows as a standard result. \square

Remark 4.2. In the applications of Lemma 4.1, the inequality (29) is often not seen in this form. For example, the function $a(\cdot)$ defined on \mathbb{Z}_+^k may satisfy that

$$|a(\mathbf{i})| \leq C(i_1 + \dots + i_{k_1})^{\alpha_1} (i_{k_1+1} + \dots + i_{k_1+k_2})^{\alpha_2},$$

for some $C > 0$, where $k_1 + k_2 = k$, and $\frac{\alpha_j}{k_j} < -\frac{1}{2}$, then it is easily verified by the arithmetic-geometric mean inequality $k^{-1} \sum_{j=1}^k y_j \geq \left(\prod_{j=1}^k y_j \right)^{1/k}$ for $y_j > 0$, that (29) is satisfied since $\alpha < 0$. It is also verified for a function $a_{\pi}(\cdot)$ which is $a(\cdot)$ with some of its variables identified.

In general when applying Lemma 4.1, we will omit the verification of (29) which usually can be easily done as indicated above. We will merely count the *total homogeneity exponents* of the bound, which in the example above is $\alpha = \alpha_1 + \alpha_2$.

For convenience, we make the following definition:

Definition 4.3. Let $X(n)$ be a stationary process with mean 0 and finite variance. We say

- $X(n)$ has a memory parameter of *at most* (denoted using \leq) H , if $\text{Var}[\sum_{n=1}^N X(n)] \leq cN^{2H}$ for some $c > 0$;
- $X(n)$ has a memory parameter (denoted using $=$) H , if $\text{Var}[\sum_{n=1}^N X(n)] \sim cN^{2H}$ as $N \rightarrow \infty$ for some $c > 0$.

Remark 4.4. In view of the definition above, Lemma 4.1 states that if $Y'(n)$ in (2) satisfies (29), then $Y'(n)$ has a memory parameter of at most $1/2$ if $H^* < 1/2$ and of at most H^* if $H^* > 1/2$.

Proposition 4.5 (Proposition 5.4 of Bai and Taqqu [2]). *Let $Y'(n)$ be given as in (27) with coefficient satisfying (29) and $H^* < 1/2$ in Lemma 4.1. Then*

$$N^{-1/2} \sum_{n=1}^{[Nt]} [Y'(n) - \mathbb{E}Y'(n)] \xrightarrow{f.d.d.} \sigma B(t),$$

where $\sigma^2 = \sum_{n=-\infty}^{\infty} \text{Cov}[Y'(n), Y'(0)]$, $B(t)$ is a standard Brownian motion, and $\xrightarrow{f.d.d.}$ stands for convergence of finite-dimensional distributions.

If each $\epsilon_i^{(1)}, \dots, \epsilon_i^{(k)}$ has a moment greater than 2, then the tightness of $N^{-1/2} \sum_{n=1}^{[Nt]} [Y'(n) - \mathbb{E}Y'(n)]$ in $D[0, 1]$ holds and thus $\xrightarrow{f.d.d.}$ can be replaced by weak convergence \Rightarrow in $D[0, 1]$.

The above $\xrightarrow{f.d.d.}$ or \Rightarrow convergence also holds for a linear combination of different $Y'(n)$'s defined on a common i.i.d. noise vector ϵ_i , while the $Y'(n)$'s can have different orders and involve different subvectors of ϵ_i , provided the coefficient of each $Y'(n)$ satisfies (29) with $H^* < 1/2$.

We now state an important result concerning the weak convergence of a discrete chaos to a Wiener chaos. Let h be a function defined in \mathbb{Z}^k such that $\sum'_{\mathbf{i} \in \mathbb{Z}_+^k} h(\mathbf{i})^2 < \infty$, where $'$ indicates the exclusion of the diagonals $i_p = i_q$, $p \neq q$. Let $Q_k(h)$ be defined as follows:

$$Q_k(h) = Q_k(h, \epsilon) = \sum_{(i_1, \dots, i_k) \in \mathbb{Z}^k} h(i_1, \dots, i_k) \epsilon_{i_1} \dots \epsilon_{i_k} = \sum_{\mathbf{i} \in \mathbb{Z}^k} h(\mathbf{i}) \prod_{p=1}^k \epsilon_{i_p}, \quad (31)$$

where ϵ_i 's are i.i.d. noise. Observe that $Q_k(h)$ is invariant under permutation of the arguments of $h(i_1, \dots, i_k)$. So if \tilde{h} is the symmetrization of h , then $Q_k(h) = Q_k(\tilde{h})$.

Suppose now that we have a sequence of function vectors $\mathbf{h}_n = (h_{1,n}, \dots, h_{j,n})$ where each $h_{j,n} \in L^2(\mathbb{Z}^{k_j})$, $j = 1, \dots, J$.

Proposition 4.6 (Proposition 4.1 of Bai and Taqqu [1]). *Let $\tilde{h}_{j,n}(\mathbf{x}) = n^{k_j/2} h_{j,n}([n\mathbf{x}] + \mathbf{c}_j)$, $j = 1, \dots, J$, where $\mathbf{c}_j \in \mathbb{Z}^{k_j}$. Suppose that there exists $h_j \in L^2(\mathbb{R}^{k_j})$, such that*

$$\|\tilde{h}_{j,n} - h_j\|_{L^2(\mathbb{R}^{k_j})} \rightarrow 0 \quad (32)$$

as $n \rightarrow \infty$. Then, as $n \rightarrow \infty$, we have the following joint convergence in distribution:

$$\mathbf{Q} := (Q_{k_1}(h_{1,n}), \dots, Q_{k_J}(h_{J,n})) \xrightarrow{d} \mathbf{I} := (I_{k_1}(h_1), \dots, I_{k_J}(h_J)).$$

5 Proof of Theorem 3.4 where diagonals are excluded

We first show that $g^{(1)} \otimes_r g^{(2)}$ in (23) is a GHK.

Lemma 5.1. Let $g^{(j)}$ be a symmetric GHK(B) with homogeneity exponent α_j defined on $\mathbb{R}_+^{k_j}$, $j = 1, 2$. Suppose in addition that either $k_1 \geq 2$ or $k_2 \geq 2$, and that

$$\alpha_1 + \alpha_2 > -(k_1 + k_2 + 1)/2, \quad (33)$$

and set

$$r = \begin{cases} 0, \dots, k_1 \wedge k_2, & \text{if } k_1 \neq k_2, \\ 0, \dots, k_1 - 1, & \text{if } k_1 = k_2. \end{cases}$$

If the function $g^{(1)} \otimes_r g^{(2)}$ is nonzero, then it is a GHK on $\mathbb{R}_+^{k_1+k_2-2r}$ with homogeneity exponent $\alpha_1 + \alpha_2 + r$.

Proof. When $r = 0$, $g^{(1)} \otimes g^{(2)}$ is a tensor product of two GHK(B)s. It is a GHK because Condition 1 of Definition 2.1 is satisfied with homogeneity exponent

$$-(k_1 + k_2 + 1)/2 < \alpha_1 + \alpha_2 < -(k_1 + k_2)/2 \quad (34)$$

(see (7)), and Condition 2 of Definition 2.1 is satisfied because

$$\int_{\mathbb{R}_+^{k_1+k_2}} |g^{(1)}(\mathbf{x}_1)g^{(2)}(\mathbf{x}_2)g^{(1)}(\mathbf{1}+\mathbf{x}_1)g^{(2)}(\mathbf{1}+\mathbf{x}_2)|d\mathbf{x}_1d\mathbf{x}_2 = \int_{\mathbb{R}_+^{k_1}} |g^{(1)}(\mathbf{x})g^{(1)}(\mathbf{1}+\mathbf{x})|d\mathbf{x} \int_{\mathbb{R}_+^{k_2}} |g^{(2)}(\mathbf{x})g^{(2)}(\mathbf{1}+\mathbf{x})|d\mathbf{x} < \infty.$$

We shall now focus on the case $r > 0$.

Consider first $k_1 \geq 2$ and $k_2 = 1$ (the case $k_1 = 1$ and $k_2 \geq 2$ is similar), so that $g^{(2)}(x) = Cx^{\alpha_2}$ for some $C \neq 0$, where $\alpha_2 \in (-1, -1/2)$. Fix an $\mathbf{x} = (x_1, \dots, x_{k_1-1}) \in \mathbb{R}_+^{k_1-1}$, then

$$\int_0^\infty |g^{(1)}(y, \mathbf{x})|y^{\alpha_2}dy \leq C \int_0^\infty (y + x_1 \dots + x_{k_1-1})^{\alpha_1}y^{\alpha_2}dy < \infty,$$

because near $y = 0$ (the other $\mathbf{x} > \mathbf{0}$), the integrand behaves like y^{α_2} , where $\alpha_2 > -1$, while near $y = \infty$, the integrand is like $y^{\alpha_1+\alpha_2}$, where $\alpha_1 < -1$ and $\alpha_2 < -1/2$. Hence $g^{(1)} \otimes_1 g^{(2)}$ is well-defined in this case. It is easy to check that $g^{(1)} \otimes_1 g^{(2)}(\lambda \mathbf{x}) = \lambda^{\alpha_1+\alpha_2+1}g^{(1)} \otimes_1 g^{(2)}(\mathbf{x})$ for any $\lambda > 0$ by using a change of variable and using the homogeneity of $g^{(j)}$. We are left to show that $g := g^{(1)} \otimes_1 g^{(2)}$ satisfies Condition 2 of Definition 2.1. This is true because the function $f(x) := \int_0^\infty (x+y)^{\alpha_1}y^{\alpha_2}dy$ is $f(x) = C_0x^{\alpha_1+\alpha_2+1}$ for some $C_0 > 0$. So

$$|g^{(1)} \otimes_1 g^{(2)}(\mathbf{x})| \leq C(x_1 + \dots + x_{k_1-1})^{\alpha_1+\alpha_2+1} =: g^*(\mathbf{x}). \quad (35)$$

for some $C > 0$. Note that $g^*(\cdot)$ is a GHK(B) on \mathbb{R}^{k_1-1} with $-(k_1-1)/2 - 1/2 < \alpha_1 + \alpha_2 + 1 < -(k_1-1)/2$ because $\alpha_1 < -1/2$, $\alpha_2 < -k_2/2$ and $\alpha_1 + \alpha_2 > -(1+k_2+1)/2$ by assumption (33). So $g = g^{(1)} \otimes_1 g^{(2)}$ satisfies Condition 2 of Definition 2.1 because the dominating function g^* does.

Suppose now that $k_1 \geq 2$ and $k_2 \geq 2$. Consider first the case $1 \leq r \leq (k_1 \wedge k_2) - 1$. Using the bound $g^{(j)}(\mathbf{x}) \leq C\|\mathbf{x}\|^{\alpha_j}$, one has by applying Cauchy-Schwartz and integrating power functions iteratively that

$$\begin{aligned} & |g^{(1)} \otimes_r g^{(2)}(\mathbf{x})| \\ & \leq C \int_{\mathbb{R}_+^r} (y_1 + \dots + y_r + x_1 + \dots + x_{k_1-r})^{\alpha_1} (y_1 + \dots + y_r + x_{k_1-r+1} + \dots + x_{k_2+k_2-2r})^{\alpha_2} dy_1 \dots dy_r \\ & = C \int_{\mathbb{R}_+^{r-1}} dy_1 \dots dy_{r-1} \left(\int_0^\infty (y_1 + \dots + y_r + x_1 + \dots + x_{k_1-r})^{2\alpha_1} dy_r \right)^{1/2} \times \\ & \quad \left(\int_0^\infty (y_1 + \dots + y_r + x_{k_1-r+1} + \dots + x_{k_2+k_2-2r})^{2\alpha_2} dy_r \right)^{1/2} \\ & \leq C \int_{\mathbb{R}_+^{r-1}} (y_1 + \dots + y_{r-1} + x_1 + \dots + x_{k_1-r})^{\alpha_1+1/2} (y_1 + \dots + y_{r-1} + x_{k_1-r+1} + \dots + x_{k_2+k_2-2r})^{\alpha_2+1/2} d\mathbf{y} \\ & \quad \dots \\ & \leq C(x_1 + \dots + x_{k_1-r})^{\alpha_1+r/2} (x_{k_1-r+1} + \dots + x_{k_1+k_2-2r})^{\alpha_2+r/2} =: g^*(\mathbf{x}). \end{aligned} \quad (36)$$

The dominating function g^* is a GHK because it is a tensor product of two GHK(B)'s on $\mathbb{R}_+^{k_j}$, $j = 1, 2$, and

$$-\frac{(k_1 - r) + (k_2 - r) + 1}{2} < (\alpha_1 + r/2) + (\alpha_2 + r/2) < -\frac{(k_1 - r) + (k_2 - r)}{2},$$

as in the inequality (34). Therefore, the bound $g^*(\mathbf{x})$ and hence the kernel $g^{(1)} \otimes_r g^{(2)}$ satisfy Condition 2 of Definition 2.1. Moreover, the homogeneity exponent of $g^{(1)} \otimes_r g^{(2)}$ is $\alpha_1 + \alpha_2 + r$ in Condition 1 of Definition 2.1. This can be easily verified as above by change of variables and using the homogeneity of $g^{(j)}$.

The only case left is: $k_1 \neq k_2 \geq 2$ and $r = k_1 \wedge k_2$. Suppose $k_1 < k_2$. In this case, Condition 2 of Definition 2.1 can be checked by first applying the iterative Cauchy-Schwartz argument leading to (36) until only one variable of $g^{(1)}$ is unintegrated, and then bounding the last fold of integration similarly as in (35). Hence in this case as well, $g^{(1)} \otimes_r g^{(2)}$ is GHK. \square

The following lemma shows a non-central convergence involving $g^{(1)} \otimes_r g^{(2)}$ in (23):

Lemma 5.2. *Suppose that all the assumptions in Lemma 5.1 hold. Let $a^j(\cdot) = g^{(j)}L^{(j)}$, $j = 1, 2$, be as assumed before. Set*

$$X'_r(n) := \sum_{(\mathbf{u}, \mathbf{i}) > \mathbf{0}}' a^{(1)}(u_1, \dots, u_r, i_1, \dots, i_{k_1-r}) a^{(2)}(u_1, \dots, u_r, i_{k_1-r+1}, \dots, i_{k_1+k_2-2r}) \epsilon_{n-i_1} \dots \epsilon_{n-i_{k_1+k_2-2r}},$$

where ϵ_i 's are i.i.d. with mean 0 and variance 1. We then have

$$\frac{1}{N^H} \sum_{n=1}^{[Nt]} X'_r(n) \xrightarrow{f.d.d.} Z_r(t) := I_{k_1+k_2-2r}(h_{t,r})$$

jointly for all the $r = 0, 1, \dots, k$ where k is as defined in Theorem 3.4, and where

$$H = \alpha_1 + \alpha_2 + (k_1 + k_2)/2 + 1 \in (1/2, 1).$$

Proof. In view of Proposition 4.6, we need only to prove the convergence for a single r and a single $t > 0$, and the joint convergence for different r 's and t 's follows. We assume for simplicity that $a^{(j)}(\cdot) = g^{(j)}(\cdot)$ (setting $L = 1$), and including a general L in (9) is easy. We focus on the case $r \geq 1$, since the case $r = 0$ follows from Theorem 6.5 of [1], although the proof for case $r = 0$ may be regarded as contained in the proof below with \mathbf{u} being an empty vector.

Let $\mathbf{u} = (u_1, \dots, u_r)$, $\mathbf{i}_1 = (i_1, \dots, i_{k_1-r})$, $\mathbf{i}_2 = (i_{k_1-r+1}, \dots, i_{k_1+k_2-2r})$, and $\mathbf{i} = (\mathbf{i}_1, \mathbf{i}_2)$. Then

$$\begin{aligned} \frac{1}{N^H} \sum_{n=1}^{[Nt]} X'_r(n) &= \sum_{\mathbf{i} \in \mathbb{Z}^{k_1+k_2-2r}}' \frac{1}{N^H} \sum_{n=1}^{[Nt]} \sum_{\mathbf{u} \in D(\mathbf{i}, n)} g^{(1)}(\mathbf{u}, n\mathbf{1} - \mathbf{i}_1) 1_{\{n\mathbf{1} > \mathbf{i}_1\}} g^{(2)}(\mathbf{u}, n\mathbf{1} - \mathbf{i}_2) 1_{\{n\mathbf{1} > \mathbf{i}_2\}} \prod_{j=1}^{k_1+k_2-r} \epsilon_{i_j} \\ &=: Q_{k_1+k_2-2r}(h_{N,t,r}), \end{aligned}$$

using the notation (31), where

$$h_{N,t,r}(\mathbf{i}) := \frac{1}{N^H} \sum_{n=1}^{[Nt]} \sum_{\mathbf{u} \in D(\mathbf{i}, n)} g^{(1)}(\mathbf{u}, n\mathbf{1} - \mathbf{i}_1) g^{(2)}(\mathbf{u}, n\mathbf{1} - \mathbf{i}_2) 1_{\{n\mathbf{1} > \mathbf{i}\}}$$

and

$$D(\mathbf{i}, n) = \{\mathbf{u} \in \mathbb{Z}_+^r : u_p \neq u_q \text{ if } p \neq q; \text{ and } u_p \neq n - i_q \text{ even if } p = q\}.$$

Set $\mathbf{x}_1 \in \mathbb{R}^{k_1-r}$, $\mathbf{x}_1 \in \mathbb{R}^{k_2-r}$ and $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$. Define

$$E(\mathbf{x}, N) = \{\mathbf{u} \in \mathbb{Z}_+^r : u_p \neq u_q \text{ if } p \neq q; \text{ and } u_p \neq n - [Nx_q] - 1 \text{ even if } p = q\}.$$

In view of Proposition 4.6 and using the homogeneity of $g^{(j)}$'s, one writes:

$$\begin{aligned}
\tilde{h}_{N,t,r}(\mathbf{x}) &= N^{(k_1+k_2-2r)/2} h_{N,t}([N\mathbf{x}] + \mathbf{1}) \\
&= \frac{1}{N^{\alpha_1+\alpha_2+r+1}} \sum_{n=1}^{[Nt]} \sum_{\mathbf{u} \in E(\mathbf{x},n)} g^{(1)}(\mathbf{u}, n\mathbf{1} - [N\mathbf{x}_1] - \mathbf{1}) g^{(2)}(\mathbf{u}, n\mathbf{1} - [N\mathbf{x}_2] - \mathbf{1}) 1_{\{n\mathbf{1} > \mathbf{i}\}} \\
&= \sum_{n=1}^{[Nt]} \frac{1}{N} \sum_{\mathbf{u} \in E(\mathbf{x},n)} \frac{1}{N^r} g^{(1)}\left(\frac{\mathbf{u}}{N}, \frac{n\mathbf{1} - [N\mathbf{x}_1] - \mathbf{1}}{N}\right) g^{(2)}\left(\frac{\mathbf{u}}{N}, \frac{n\mathbf{1} - [N\mathbf{x}_2] - \mathbf{1}}{N}\right) 1_{\{n\mathbf{1} > \mathbf{i}\}} \\
&= \int_0^t ds \int_{\mathbb{R}_+^r} d\mathbf{y} g^{(1)}\left(\frac{[N\mathbf{y}] + \mathbf{1}}{N}, \frac{[Ns]\mathbf{1} - [N\mathbf{x}_1]}{N}\right) g^{(2)}\left(\frac{[N\mathbf{y}] + \mathbf{1}}{N}, \frac{[Ns]\mathbf{1} - [N\mathbf{x}_2]}{N}\right) 1_{\{[Ns]\mathbf{1} > [N\mathbf{x}]\} \cap F(N)},
\end{aligned}$$

where we correspond \mathbf{u} to $[N\mathbf{y}] + \mathbf{1}$, n to $[Ns] + 1$, and

$$F(N) = \{(\mathbf{x}, \mathbf{y}, s) : [Ny_p] \neq [Ny_q], [Nx_p] \neq [Nx_q], \text{ if } p \neq q; \text{ and } [Ny_p] \neq [Ns] - [Nx_q] \text{ even if } p = q\}.$$

In view of Proposition 4.6, the goal is to show that

$$\lim_{N \rightarrow \infty} \|\tilde{h}_{N,t,r} - h_{t,r}\|_{L^2(\mathbb{R}^{k_1+k_2-2r})} = 0. \quad (37)$$

where $h_{t,r}$ is given in (22). By the a.e. continuity of $g^{(j)}$'s and the fact that $1_{F(N)} \rightarrow 1$ a.e. as $N \rightarrow \infty$, one has

$$\begin{aligned}
&g^{(1)}\left(\frac{[N\mathbf{y}] + \mathbf{1}}{N}, \frac{[Ns]\mathbf{1} - [N\mathbf{x}_1]}{N}\right) g^{(2)}\left(\frac{[N\mathbf{y}] + \mathbf{1}}{N}, \frac{[Ns]\mathbf{1} - [N\mathbf{x}_2]}{N}\right) 1_{\{[Ns]\mathbf{1} > [N\mathbf{x}]\} \cap F(N)} \rightarrow \\
&g^{(1)}(\mathbf{y}, s\mathbf{1} - \mathbf{x}_1) g^{(2)}(\mathbf{y}, s\mathbf{1} - \mathbf{x}_2) 1_{\{s\mathbf{1} > \mathbf{x}\}} \quad \text{for a.e. } (\mathbf{x}, \mathbf{y}, s).
\end{aligned}$$

We are left to establish suitable bound to apply the Dominated Convergence Theorem. To this end, since $g^{(j)}(\mathbf{x}) \leq C\|\mathbf{x}\|^{\alpha_j} =: g^{(j)*}(\mathbf{x})$ on $\mathbb{R}_+^{k_j}$, we have the following bound

$$\begin{aligned}
&\left| g^{(1)}\left(\frac{[N\mathbf{y}] + \mathbf{1}}{N}, \frac{[Ns]\mathbf{1} - [N\mathbf{x}_1]}{N}\right) g^{(2)}\left(\frac{[N\mathbf{y}] + \mathbf{1}}{N}, \frac{[Ns]\mathbf{1} - [N\mathbf{x}_2]}{N}\right) \right| 1_{\{[Ns]\mathbf{1} > [N\mathbf{x}]\} \cap F(N)} \\
&\leq g^{(1)*}\left(\frac{[N\mathbf{y}] + \mathbf{1}}{N}, \frac{[Ns]\mathbf{1} - [N\mathbf{x}_1]}{N}\right) g^{(2)*}\left(\frac{[N\mathbf{y}] + \mathbf{1}}{N}, \frac{[Ns]\mathbf{1} - [N\mathbf{x}_2]}{N}\right) 1_{\{[Ns]\mathbf{1} > [N\mathbf{x}]\} \cap F(N)} \\
&\leq C g^{(1)*}(\mathbf{y}, s\mathbf{1} - \mathbf{x}_1) g^{(2)*}(\mathbf{y}, s\mathbf{1} - \mathbf{x}_2) 1_{\{s\mathbf{1} > \mathbf{x}\}},
\end{aligned} \quad (38)$$

where we have used the following facts: on the set $\{\mathbf{y} > \mathbf{0}, [Ns]\mathbf{1} > [N\mathbf{x}]\}$, we have $([N\mathbf{y}] + 1)/N > \mathbf{y}$, $([Ns] - [Nx_j])/N \geq \frac{1}{2}(s - x_j)$ (see Relation (40) in the proof of Theorem 6.5 of Bai and Taqqu [1]) and $g^{(j)*}$ decreases in its every variable, as well as the fact that $\{[Ns]\mathbf{1} > [N\mathbf{x}]\} \subset \{s\mathbf{1} > \mathbf{x}\}$. Note that

$$\int_0^t ds \int_{\mathbb{R}_+^r} d\mathbf{y} g^{(1)*}(\mathbf{y}, s\mathbf{1} - \mathbf{x}_1) g^{(2)*}(\mathbf{y}, s\mathbf{1} - \mathbf{x}_2) 1_{\{s\mathbf{1} > \mathbf{x}\}} = \int_0^t g^{(1)*} \otimes_r g^{(2)*}(s\mathbf{1} - \mathbf{x}) 1_{\{s\mathbf{1} > \mathbf{x}\}} ds. \quad (39)$$

Since $g^{(1)*}$ and $g^{(2)*}$ are GHK(B)'s, so by Lemma 5.1, $g^{(1)*} \otimes_r g^{(2)*}$ is a GHK. This has two consequences. First, by Theorem 3.5 and Remark 3.6 of Bai and Taqqu [1], the integral in $ds d\mathbf{y}$ on the left-hand side of (39) is finite for a.e. $\mathbf{x} \in \mathbb{R}^{k_1+k_2-2r}$. One can then apply the Dominated Convergence Theorem to conclude that

$$\tilde{h}_{N,t,r}(\mathbf{x}) \rightarrow h_{t,r}(\mathbf{x}) \text{ for a.e. } \mathbf{x} \in \mathbb{R}^{k_1+k_2-2r}. \quad (40)$$

But to obtain (37), we need L^2 convergence for the integral in $d\mathbf{x}$. For this, we use the bound (38):

$$|\tilde{h}_{N,t,r}(\mathbf{x})| \leq h_{t,r}^*(\mathbf{x}) := C \int_0^t g^{(1)*} \otimes_r g^{(2)*}(s\mathbf{1} - \mathbf{x}) 1_{\{s\mathbf{1} > \mathbf{x}\}} ds.$$

The second consequence of the fact that $g^{(1)*} \otimes_r g^{(2)*}$ is a GHK stems from Remark 2.2, which entails that $h_{t,r}^* \in L^2(\mathbb{R}^{k_1+k_2-2r})$, and hence (37) follows from (40) and the Dominated Convergence Theorem. This concludes the proof of Lemma 5.2. \square

We now decompose the product $X(n)$ in (20) in off-diagonal forms (27) as follows: let $\mathbf{u} = (u_1, \dots, u_r) \in \mathbb{Z}_+^r$, $\mathbf{i}_1 = (i_1, \dots, i_{k_1-r})$ and $\mathbf{i}_2 = (i_{k_1-r+1}, \dots, i_{k_1+k_2-2r})$, and $\mathbf{i} = (\mathbf{i}_1, \mathbf{i}_2) \in \mathbb{Z}_+^{k_1+k_2-2r}$, then

$$X(n) = Y_1'(n)Y_2'(n) = \sum_{r=0}^{k_1 \wedge k_2} r! \binom{k_1}{r} \binom{k_2}{r} \sum_{(\mathbf{u}, \mathbf{i}) \in \mathbb{Z}_+^{k_1+k_2-2r}} a^{(1)}(\mathbf{u}, \mathbf{i}_1) a^{(2)}(\mathbf{u}, \mathbf{i}_2) \epsilon_{n-u_1}^2 \dots \epsilon_{n-u_r}^2 \epsilon_{n-i_1} \dots \epsilon_{n-i_{k_1+k_2-2r}}.$$

where we have used the symmetry of $a^{(j)}$'s, while the combinatorial coefficient

$$c(r, k_1, k_2) := r! \binom{k_1}{r} \binom{k_2}{r}$$

is obtained as the number of ways to pair r variables of $a^{(1)}$ to r variables of $a^{(2)}$. We write

$$\epsilon_{n-i}^2 = 1 + (\epsilon_{n-i}^2 - 1) =: A_0(\epsilon_{n-i}) + A_2(\epsilon_{n-i}),$$

where $A_0(\epsilon) = 1$ and $A_2(\epsilon) = \epsilon^2 - 1$. These are Appell polynomials which will be introduced more fully in Section 6. Set $J_r = \{0, 2\} \times \dots \times \{0, 2\}$. Then

$$Y_1'(n)Y_2'(n) = \sum_{r=0}^{k_1 \wedge k_2} c(r, k_1, k_2) \sum_{(\mathbf{u}, \mathbf{i}) \in \mathbb{Z}_+^{k_1+k_2-2r}} \sum_{\mathbf{j} \in J_r} a^{(1)}(\mathbf{u}, \mathbf{i}_1) a^{(2)}(\mathbf{u}, \mathbf{i}_2) A_{j_1}(\epsilon_{n-u_1}) \dots A_{j_r}(\epsilon_{n-u_r}) \epsilon_{n-i_1} \dots \epsilon_{n-i_{k_1+k_2-2r}}.$$

The random variables in each summand are independent because the sum does not include diagonals. Observe that it is only when $k_1 = k_2$, that the mean

$$\mathbb{E}Y_1'(n)Y_2'(n) = k_1! \sum_{\mathbf{u} \in \mathbb{Z}_+^{k_1}} a^{(1)}(\mathbf{u}) a^{(2)}(\mathbf{u})$$

may possibly be nonzero (this is the case when $r = k_1 = k_2$). Hence one can use the k defined in Theorem 3.4 to write that

$$X(n) - \mathbb{E}X(n) = \sum_{r=0}^k \sum_{\mathbf{j} \in J_r} \sum_{(\mathbf{u}, \mathbf{i}) \in \mathbb{Z}_+^{k_1+k_2-2r}} c(r, k_1, k_2) a^{(1)}(\mathbf{u}, \mathbf{i}_1) a^{(2)}(\mathbf{u}, \mathbf{i}_2) A_{j_1}(\epsilon_{n-u_1}) \dots A_{j_r}(\epsilon_{n-u_r}) \epsilon_{n-i_1} \dots \epsilon_{n-i_{k_1+k_2-2r}}. \quad (41)$$

A basic term of the preceding decomposition of $X(n) - \mathbb{E}X(n)$ is

$$X_{\mathbf{j}}^r(n) := \sum_{(\mathbf{u}, \mathbf{i}) \in \mathbb{Z}_+^{k_1+k_2-2r}} c(r, k_1, k_2) a^{(1)}(\mathbf{u}, \mathbf{i}_1) a^{(2)}(\mathbf{u}, \mathbf{i}_2) A_{j_1}(\epsilon_{n-u_1}) \dots A_{j_r}(\epsilon_{n-u_r}) \epsilon_{n-i_1} \dots \epsilon_{n-i_{k_1+k_2-2r}}.$$

Note that $0 \leq r \leq k_1 \wedge k_2$ if $k_1 \neq k_2$, and $0 \leq r \leq k_1 - 1$ if $k_1 = k_2$, which implies $k_1 + k_2 - 2r \geq 1$ so that there is at least one i variable. Due to the symmetry of $a^{(j)}$'s, we can suppose without loss of generality that $j_1 = \dots j_s = 0$ and $j_{s+1} = \dots j_r = 2$, $0 \leq s \leq r$. One can hence rewrite the basic term as:

$$X_{\mathbf{j}}^r(n) = \sum_{(\mathbf{u}, \mathbf{i}) \in \mathbb{Z}_+^{k_1+k_2-2r}} c(r, k_1, k_2) a^{(1)}(\mathbf{u}, \mathbf{i}_1, \mathbf{i}_2) a^{(2)}(\mathbf{u}, \mathbf{i}_1, \mathbf{i}_3) A_2(\epsilon_{n-i_1}) \dots A_2(\epsilon_{n-i_{r-s}}) \epsilon_{n-i_{r-s+1}} \dots \epsilon_{n-i_{k_1+k_2-2r-s}},$$

where $\mathbf{u} = (u_1, \dots, u_s)$, $\mathbf{i}_1 = (i_1, \dots, i_{r-s})$, $\mathbf{i}_2 = (i_{r-s+1}, \dots, i_{k_1-s})$ and $\mathbf{i}_3 = (i_{k_1-s+1}, \dots, i_{k_1+k_2-r-s})$ and $\mathbf{i} = (\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3)$. Setting

$$a'(\mathbf{i}) = \sum_{\mathbf{u} \in K(\mathbf{i})} c(r, k_1, k_2) a^{(1)}(\mathbf{u}, \mathbf{i}_1, \mathbf{i}_2) a^{(2)}(\mathbf{u}, \mathbf{i}_1, \mathbf{i}_3), \quad (42)$$

with

$$K(\mathbf{i}) = \{\mathbf{u} > \mathbf{0} : u_p \neq u_q \text{ if } p \neq q; \text{ and } u_p \neq i_q \text{ even if } p = q\},$$

we get

$$X_{\mathbf{j}}^r(n) = \sum_{\mathbf{i} > \mathbf{0}}^r a'(\mathbf{i}) A_2(\epsilon_{n-i_1}) \dots A_2(\epsilon_{n-i_{r-s}}) \epsilon_{n-i_{r-s+1}} \dots \epsilon_{n-i_{k_1+k_2-r-s}}. \quad (43)$$

We list here some useful elementary inequalities which will be used many times in the sequel:

Lemma 5.3. *Let $A > 0$, $B > 0$.*

If $\gamma < -1$, then

$$\sum_{i=1}^{\infty} (A+i)^{\gamma} \leq CA^{\gamma+1}. \quad (44)$$

If $\gamma < 0$, $\beta < -1$, then

$$\sum_{i=1}^{\infty} (A+i)^{\gamma} i^{\beta} \leq CA^{\gamma}. \quad (45)$$

If $\gamma < -1/2$, $-1 < \beta < -1/2$, then

$$\sum_{i=1}^{\infty} (A+i)^{\gamma} i^{\beta} \leq CA^{\gamma+\beta+1}. \quad (46)$$

If $\gamma < -1/2$, $\beta < -1/2$, then

$$\sum_{i=1}^{\infty} (A+i)^{\gamma} (B+i)^{\beta} \leq CA^{\gamma+1/2} B^{\beta+1/2}. \quad (47)$$

Proof. To obtain inequality (44), we have

$$\sum_{i=1}^{\infty} (A+i)^{\gamma} = \sum_{i=1}^{\infty} \int_{i-1}^i (A+i)^{\gamma} dx \leq \sum_{i=1}^{\infty} \int_{i-1}^i (A+x)^{\gamma} dx = \int_0^{\infty} (A+x)^{\gamma} dx = -(\gamma+1)^{-1} A^{\gamma+1}.$$

For (45), note that $(A+i)^{\gamma} \leq A^{\gamma}$ and $\sum_{i=1}^{\infty} i^{\beta} < \infty$.

For inequality (46), we have

$$\sum_{i=1}^{\infty} (A+i)^{\gamma} i^{\beta} = A^{\gamma+\beta+1} \sum_{i=1}^{\infty} \int_{i-1}^i (1+i/A)^{\gamma} (i/A)^{\beta} d(x/A) \leq A^{\gamma+\beta+1} \int_0^{\infty} (1+y)^{\gamma} y^{\beta} dy,$$

where the integral is finite since $\beta > -1$ and $\gamma + \beta < -1$.

The last one (47) is obtained by applying Cauchy-Schwartz and (44) as follows:

$$\sum_{i=1}^{\infty} (A+i)^{\gamma} (B+i)^{\beta} \leq \left[\sum_{i=1}^{\infty} (A+i)^{2\gamma} \right]^{1/2} \left[\sum_{i=1}^{\infty} (B+i)^{2\beta} \right]^{1/2} \leq CA^{\gamma+1/2} B^{\beta+1/2}.$$

□

Remark 5.4. The inequalities (44) (46) and (47) all raise the total power exponent by 1, while inequality (45) kills one of the exponents. These observations are useful in the proof below and also in Section 6.

We now state the proof of Theorem 3.4:

Proof of Case 1 of Theorem 3.4. We want to apply Proposition 4.5. The condition $\mathbb{E}|\epsilon_i|^{4+\delta} < \infty$ guarantees that $\mathbb{E}|A_2(\epsilon)|^{2+\delta'} < \infty$ in (43) holds for some $\delta' > 0$ and so the tightness in $D[0, 1]$ holds.

We only need to show that $H^* < 1/2$ in Lemma 4.1 for each of the basic term $X_{\mathbf{j}}^r(n)$ in (43).

Suppose without loss of generality that $k_1 \leq k_2$. Using the fact $|a^{(j)}(\mathbf{i})| \leq C\|\mathbf{i}\|^{\alpha_j}$ (recall that $\|\cdot\|$ is the L^1 -norm), one can bound $a'(\mathbf{i})$ in (42). One has to distinguish two cases. In the first case, where $s < k_1$, one gets

$$\begin{aligned} |a'(\mathbf{i})| &\leq C \sum_{\mathbf{u} \in \mathbb{Z}_+^s} \|(\mathbf{u}, \mathbf{i}_1, \mathbf{i}_2)\|^{\alpha_1} \|(\mathbf{u}, \mathbf{i}_1, \mathbf{i}_3)\|^{\alpha_2} \\ &\leq C \sum_{\mathbf{u} \in \mathbb{Z}_+^s} (u_1 + \dots + u_s + \|\mathbf{i}_1\| + \|\mathbf{i}_2\|)^{\alpha_1} (u_1 + \dots + u_s + \|\mathbf{i}_1\| + \|\mathbf{i}_3\|)^{\alpha_2} \\ &\leq C(\|\mathbf{i}_1\| + \|\mathbf{i}_2\|)^{\alpha_1 + s/2} (\|\mathbf{i}_1\| + \|\mathbf{i}_3\|)^{\alpha_2 + s/2}, \end{aligned}$$

after applying (47) to each of the s components of \mathbf{u} iteratively (note: \mathbf{i}_1 may not be present). In the second case, where $s = r = k_1$, one gets

$$\begin{aligned} |a'(\mathbf{i})| &\leq C \sum_{\mathbf{u} \in \mathbb{Z}_+^s} \|\mathbf{u}\|^{\alpha_1} \|(\mathbf{u}, \mathbf{i}_3)\|^{\alpha_2} \\ &\leq C \sum_{\mathbf{u} \in \mathbb{Z}_+^s} (u_1 + \dots + u_s)^{\alpha_1} (u_1 + \dots + u_s + \|\mathbf{i}_3\|)^{\alpha_2} \\ &\leq C\|\mathbf{i}\|^{\alpha_1 + \alpha_2 + s}, \end{aligned}$$

after applying (47) $s - 1$ times, and then (46) to the last component of \mathbf{u} . In either case, the total power exponent is raised by s .

As in (30), this yields an

$$\begin{aligned} H^* &= \alpha_1 + \alpha_2 + s + (r - s + k_1 - r + k_2 - r)/2 + 1 \\ &= H_1 + H_2 + (s - r)/2 - 1 \\ &\leq H_1 + H_2 - 1 < 1/2, \end{aligned} \tag{48}$$

where the last strict equality is due to the assumption $H_1 + H_2 < 3/2$ of Case 1. □

Proof of Case 2 of Theorem 3.4. We now suppose that $H_1 + H_2 > 3/2$. As was shown in Case 1 above, the off-diagonal chaos coefficient $a'(\cdot)$ in (42) leads to

$$H^* = H_1 + H_2 + (s - r)/2 - 1.$$

When $s = r$, we have only factors $A_0(\epsilon) = 1$ in (43). The chaos process $X_r^j(n)$ is up to some constant the process $X'_r(n)$ in Lemma 5.2. So collecting all these terms for different r in (41), one obtains the non-central limit claimed in the theorem with an Hurst index $H = H_1 + H_2 - 1 > 1/2$. Tightness in $D[0, 1]$ is automatic since $H > 1/2$ (see, e.g. Proposition 4.4.2 of Giraitis et al. [6]).

When $s < r$, the corresponding terms are negligible. Indeed,

$$H^* = H_1 + H_2 + (s - r)/2 - 1 \leq H_1 + H_2 - 1/2 - 1 < 1/2.$$

So by Lemma 4.1, the term $X_r^j(n)$ has a memory parameter $H \leq 1/2$ in the sense of Definition 4.3. Hence $N^{-(H_1 + H_2 - 1)} \sum_{n=1}^{[Nt]} X_r^j(n)$ converges in probability to 0 in $D[0, 1]$. □

6 Proof of Theorem 3.5 where diagonals are included

We first recall from Bai and Taqqu [2] the off-diagonal decomposition of a general k -th order Volterra process $X(n)$ in (4). The purpose is to decompose $X(n)$ into off-diagonal chaos terms as in (27). To this end, it is

convenient to use Appell polynomials. Suppose that ϵ is a random variable with finite K -th moment. The Appell polynomial with respect to the law of ϵ is defined through the following recursive relation:

$$\frac{d}{dx}A_p(x) = pA_{p-1}, \quad \mathbb{E}A_p(\epsilon) = 0, \quad A_0(x) = 1, \quad p = 1, \dots, K.$$

We will use the following identity:

$$x^p = \sum_{j=0}^p \binom{p}{j} \mu_{p-j} A_j(x), \quad p = 0, 1, 2, 3, \dots \quad (49)$$

For more details about Appell polynomials, see for example Chapter 3.3 of Beran et al. [3].

Let \mathcal{P}_k be the collection of all the partitions of $\{1, \dots, k\}$. We further express each partition $\pi \in \mathcal{P}_k$ as $\pi = (P_1, \dots, P_m)$ (so $m = |\pi|$), where the sets P_t 's are *ordered* according to their smallest element. If we have a variable $\mathbf{i} \in \mathbb{Z}_+^k$, then \mathbf{i}_π denotes a new variable where its components are identified according to π . For example, if $k = 3$, $\pi = (\{1, 2\}, \{3\})$ and $\mathbf{i} = (i_1, i_2, i_3)$, then $\mathbf{i}_\pi = (i_1, i_1, i_2)$. In this case we write $\pi = (P_1, P_2)$ where $P_1 = \{1, 2\}$ and $P_2 = \{3\}$. If $a(\cdot)$ is a function on \mathbb{Z}_+^k , then

$$a_\pi(i_1, \dots, i_m) := a(\mathbf{i}_\pi), \quad (50)$$

where $m = |\pi|$. In the preceding example, $a_\pi(\mathbf{i}) = a(i_1, i_2, i_2)$ with $m = 2$. We define a summation operator S'_T as follows: for any $T \subset \{1, \dots, |\pi|\}$, $S'_T(a_\pi)$ is obtained by summing a_π over its variables indicated by T off-diagonally, yielding a function with $|\pi| - |T|$ variables. For instance, if $\pi = (\{1, 5\}, \{2\}, \{3, 4\})$, then $\mathbf{i}_\pi = (i_1, i_2, i_3, i_3, i_1)$ and if $T = \{1, 3\}$, then

$$(S'_T a_\pi)(i) = \sum_{0 < i_1, i_3 < \infty}^I a(i_1, i, i_3, i_3, i_1),$$

provided that it is well-defined. Note that in this off-diagonal sum, we require also that neither i_1 nor i_3 equals to i . If $T = \emptyset$, S'_T is understood to be the identity operator.

Now, by collecting various diagonal cases and using (49), $X(n)$ in (4) can be decomposed as

$$X(n) = \sum_{\pi \in \mathcal{P}_k} \sum_{\mathbf{i} \in \mathbb{Z}_+^m}^I a_\pi(\mathbf{i}) \epsilon_{n-i_1}^{p_1} \dots \epsilon_{n-i_m}^{p_m} = \sum_{\pi \in \mathcal{P}_k} \sum_{\mathbf{j} \in J(\pi)} X_\pi^{\mathbf{j}}(n) \quad (51)$$

where

$$X_\pi^{\mathbf{j}}(n) = \sum_{\mathbf{i} \in \mathbb{Z}_+^m}^I a_\pi(\mathbf{i}) c(\mathbf{p}, \mathbf{j}) A_{j_1}(\epsilon_{n-i_1}) \dots A_{j_m}(\epsilon_{n-i_m}),$$

$A_j(\cdot)$ is the j -th order Appell polynomial with respect to the law of ϵ_i , $p_t = |P_t|$, $J(\pi) = \{0, \dots, p_1\} \times \dots \times \{0, \dots, p_m\}$, and

$$c(\mathbf{p}, \mathbf{j}) = \binom{p_1}{j_1} \dots \binom{p_m}{j_m} \mu_{p_1-j_1} \dots \mu_{p_m-j_m}, \quad \mu_j = \mathbb{E}\epsilon_i^j. \quad (52)$$

Note that since by assumption $\mu_1 = 0$, when $j_t = 0$, it is only when $p_t \geq 2$ that it is possible to have a nonzero term.

In addition, the expression for the centered $X(n) - \mathbb{E}X(n)$ is the sum in (51) with $J(\pi)$ replaced by $J^+(\pi) := J(\pi) \setminus (0, \dots, 0)$, and

$$\mathbb{E}X(n) = \sum_{\pi \in \mathcal{P}_k} \sum_{\mathbf{i} \in \mathbb{Z}_+^m}^I a_\pi(\mathbf{i}) \mu_{p_1} \dots \mu_{p_m} = \sum_{\pi \in \mathcal{P}_k^2} \sum_{\mathbf{i} \in \mathbb{Z}_+^m}^I a_\pi(\mathbf{i}) \mu_{p_1} \dots \mu_{p_m},$$

where \mathcal{P}_k^2 denotes the collection of partitions of $\{1, \dots, k\}$ such that each set in the partition contains at least 2 elements, namely, $p_t \geq 2$ for all $t = 1, \dots, m$.

One basic term of the off-diagonal decomposition of $X(n) - \mathbb{E}X(n)$ is

$$X_\pi^{\mathbf{j}}(n) = \sum_{\mathbf{i} \in \mathbb{Z}_+^m} a_\pi(\mathbf{i}) c(\mathbf{p}, \mathbf{j}) A_{j_1}(\epsilon_{n-i_1}) \dots A_{j_m}(\epsilon_{n-i_m}) = \sum_{\mathbf{i} \in \mathbb{Z}_+^{k'}} c(\mathbf{p}, \mathbf{j}) S'_T a_\pi(\mathbf{i}) A_{j_{t_1}}(\epsilon_{n-i_{t_1}}) \dots A_{j_{t_{k'}}}(\epsilon_{n-i_{t_{k'}}}) \quad (53)$$

where $T = \{t = 1, \dots, m : j_t = 0\}$, and $\{t_1, \dots, t_{k'}\} = \{1, \dots, m\} \setminus T$ (thus $j_{t_1} \geq 1, \dots, j_{t_{k'}} \geq 1$). Note that $T \neq \{1, \dots, m\}$ since $\mathbf{j} \in J^+(\pi)$. In fact, $X_\pi^{\mathbf{j}}(n)$ is of the form (27) with $k = k'$ and $a(\cdot) = c(\mathbf{p}, \mathbf{j}) S'_T a_\pi(\cdot)$.

We now state the proof of Theorem 3.5 case by case. Recall that $C > 0$ denotes a constant whose value can change from line to line.

Proof of Case 1. In this case, $g^{(1)}(i) = C_1 i^{\alpha_1}$, and $g^{(2)}(i) = C_2 i^{\alpha_2}$, where C_1 and C_2 are two nonzero constants. The off-diagonal decomposition (51) for the centered $X(n)$ is simply

$$X(n) - \mathbb{E}X(n) = \sum_{0 < i_1, i_2 < \infty} a^{(1)}(i_1) a^{(2)}(i_2) \epsilon_{n-i_1} \epsilon_{n-i_2} + \sum_{0 < i < \infty} a^{(1)}(i) a^{(2)}(i) A_2(\epsilon_{n-i}), \quad (54)$$

where $A_2(\epsilon_{n-i}) = \epsilon_{n-i}^2 - 1$. Note that

$$|a^{(1)}(i_1) a^{(2)}(i_2)| \leq C i_1^{\alpha_1} i_2^{\alpha_2},$$

so the coefficient of first term in (54) satisfies (29) with

$$H^* = \alpha_1 + \alpha_2 + (1+1)/2 + 1 = (H_1 - 3/2) + (H_2 - 3/2) + 2 < 1/2$$

by (19), since $H_1 + H_2 < 3/2$. For the second term in (29), one has

$$|a^{(1)}(i) a^{(2)}(i)| \leq C i^{\alpha_1 + \alpha_2},$$

which yields an

$$H^* = \alpha_1 + \alpha_2 + 1/2 + 1 = (H_1 - 3/2) + (H_2 - 3/2) + 3/2 = H_1 + H_2 - 3/2 < 1/2, \quad (55)$$

since $H_1 < 1$ and $H_2 < 1$. Hence Proposition 4.5 applies. \square

Proof of Case 2. Now the first term of (54) is subject to Proposition 2.5 with an Hurst index $H = \alpha_1 + \alpha_2 + 2 = H_1 + H_2 - 1 > 1/2$. One can see that for the second term of (54), relation (55) still holds. So by Lemma 4.1, the second term of (54) has a memory parameter $H \leq 1/2$ in the sense of Definition 4.3, and hence with the normalization N^{-H} , the normalized partial sum of the second term of (54) converges to 0 in $D[0, 1]$. \square

Proof of Case 3. Recall from (53) that a basic term of the off-diagonal decomposition is

$$X_\pi^{\mathbf{j}}(n) = \sum_{\mathbf{i} \in \mathbb{Z}_+^{k'}} c(\mathbf{p}, \mathbf{j}) S'_T a_\pi(\mathbf{i}) A_{j_{t_1}}(\epsilon_{n-i_{t_1}}) \dots A_{j_{t_{k'}}}(\epsilon_{n-i_{t_{k'}}}).$$

Consider first the following partition $\pi = (P_1, \dots, P_m)$ of $\{1, \dots, k_1, k_1 + 1\}$ which we express as $\pi = (P_1, \dots, P_{m_1}, \{k_1 + 1\})$, with $m_1 = m - 1$, $\cup_{j=1}^{m_1} P_j = \{1, \dots, k_1\}$, and $P_m = \{k_1 + 1\}$. Let $T = \{1, \dots, m_1\}$. Recall that to have nonzero $c(\mathbf{p}, \mathbf{j})$, one must require $|P_t| \geq 2$ if $t \in T$, and hence $2m_1 \leq k_1$. Set $\pi_1 = \{P_1, \dots, P_{m_1}\}$ and let $\mathbf{u} \in \mathbb{Z}_+^{k_1}$. Then applying the off-diagonal summation S'_T , we get

$$(S'_T a_\pi)(i) = \sum_{u_p \neq u_q, u_p \neq i} a_{\pi_1}^{(1)}(\mathbf{u}) a^{(2)}(i) = \left(\sum_{u_p \neq u_q} a_{\pi_1}^{(1)}(\mathbf{u}) \right) a^{(2)}(i) - R(i), \quad (56)$$

where the difference $R(i)$ includes the terms where some $u_p = i$. Since $|a^{(1)}(\mathbf{i})| \leq C(i_1 + \dots + i_{k_1})^{\alpha_1}$ which implies $|a_{\pi_1}^{(1)}(\mathbf{u})| \leq C(u_1 + \dots + u_{m_1})^{\alpha_1}$. Suppose without loss of generality that $u_{m_1} = i$, then by applying (44),

$$|R(i)| \leq C \sum_{0 < u_1, \dots, u_{m_1-1} < \infty} (u_1 + \dots + u_{m_1-1} + i)^{\alpha_1} i^{\alpha_2} \leq C i^{\alpha_2 + (\alpha_1 + m_1 - 1)},$$

where $\alpha_1 + m_1 - 1 < 0$ because $\alpha_1 < -k_1/2 \leq -m_1 \leq -1$. It follows that $|R(i)| \leq C i^{\alpha_2 - \delta}$ for some $\delta > 0$. Since $k_2 = 1$, the term $R(i)$ defines the linear process $\sum_{i>0} R(i) \epsilon_{n-i}$ but one with smaller memory parameter in the sense of Definition 4.3, than the linear process:

$$\mu_{\pi_1} \left(\sum_{\mathbf{u}>\mathbf{0}}' a_{\pi_1}^{(1)}(\mathbf{u}) \right) \sum_{i=1}^{\infty} a^{(2)}(i) \epsilon_{n-i}.$$

resulting from the first term in the right-hand side of (56) (in this case $c(\mathbf{p}, \mathbf{j}) = \mu_{\pi_1} := \mu_{p_1} \dots \mu_{p_{m_1}}$). Collecting all such $\pi_1 \in \mathcal{C}_1^2$, one obtains $c_1 \sum_{i=1}^{\infty} a^{(2)}(i) \epsilon_{n-i}$ with c_1 as given in (26). Applying Proposition 2.6 with $k = 1$, we get the non-central limit in Case 3, with a Hurst index $H = \alpha_2 + 1/2 + 1 = \alpha_2 + 3/2 = H_2$.

We now show that in all the other cases, the memory parameter of $X_{\pi}^{\mathbf{j}}(n)$ is smaller than $H = \alpha_2 + 3/2$, which will conclude the proof. Observe first that

$$|a(\mathbf{i})| \leq C(i_1 + \dots + i_{k_1})^{\alpha_1} i_{k_1+1}^{\alpha_2}. \quad (57)$$

Let $\pi = \{P_1, \dots, P_m\}$ is a partition of $\{1, \dots, k_1 + 1\}$, and $T = \{t_1, \dots, t_l\}$, $l \leq m - 1$. To bound $|(S'_T a)(\mathbf{i})|$, one can assume without loss of generality that either

- (a) $P_j \cap \{k_1 + 1\} = \emptyset$ for $1 \leq j \leq m - 1$, $P_m = \{k_1 + 1\}$, $T \subset \{1, \dots, m - 1\}$, $\cup_{j=1}^l P_{t_j} \neq \{1, \dots, k_1\}$, or
- (b) $P_m \cap \{k_1 + 1\} \neq \emptyset$, and $P_m \cap \{1, \dots, k_1\} \neq \emptyset$.

Observe that in the previous case we had $\cup_{j=1}^l P_{t_j} = \{1, \dots, k_1\}$ ($l = m_1 = m - 1$) and $P_m = \{k_1 + 1\}$.

In case (a), one has by (57) that

$$|a_{\pi}(\mathbf{i})| \leq C(i_1 + \dots + i_{m-1})^{\alpha_1} i_m^{\alpha_2}.$$

Since in case (a), $\cup_{j=1}^l P_{t_j}$ is a strict subset of $\{1, \dots, k_1\}$, we have $l < m - 1$, and thus by applying (44) iteratively, one has that:

$$|(S'_T a_{\pi})(\mathbf{i})| \leq \sum_{\mathbf{u}>\mathbf{0}} C(u_1 + \dots + u_l + i_1 + \dots + i_{m-l-1})^{\alpha_1} i_{m-l}^{\alpha_2} \leq C(i_1 + \dots + i_{m-l-1})^{\alpha_1 + l} i_{m-l}^{\alpha_2},$$

which results in an H^* in (30) equal to

$$\begin{aligned} H^* &= (\alpha_1 + l + \alpha_2) + (m - l)/2 + 1 = \alpha_1 + \alpha_2 + m/2 + l/2 + 1 \\ &< -k_1/2 + \alpha_2 + (k_1 + 1)/2 + 1 = \alpha_2 + 3/2 = H_2. \end{aligned}$$

since $\alpha_1 < -k_1/2$, and $m + l = 2l + (m - l) \leq k_1 + 1$ (recall that each $|P_t| \geq 2$ if $t \in T$).

In case (b), one can write without loss of generality that

$$|a_{\pi}(\mathbf{i})| \leq C(i_1 + \dots + i_m)^{\alpha_1} i_1^{\alpha_2}$$

since π contains m partitions. If for the above a_{π} , the summation S'_T includes a sum over the index 1, that is, $1 \in T$, then using (44) and then (46), one has

$$\begin{aligned} |(S'_T a_{\pi})(\mathbf{i})| &\leq C \sum_{\mathbf{u}>\mathbf{0}} (u_1 + \dots + u_l + i_1 + \dots + i_{m-l})^{\alpha_1} u_1^{\alpha_2} \\ &\leq C \sum_{u_1=1}^{\infty} (u_1 + i_1 + \dots + i_{m-l})^{\alpha_1 + l - 1} u_1^{\alpha_2} \leq C(i_1 + \dots + i_{m-l})^{\alpha_1 + \alpha_2 + l}. \end{aligned}$$

Relation (46) does apply because on one hand $\alpha_2 > -1$, and on the other hand, we have $\alpha_1 + l - 1 < -1/2$ since $\alpha_1 < -k_1/2$ and $2(l - 1) + 1 < k_1$ because of $|P_t| \geq 2$ if $t \in T$. This leads to an H^* in (30) equals to

$$H^* = (\alpha_1 + \alpha_2 + l) + (m - l)/2 + 1 = \alpha_1 + \alpha_2 + m/2 + l/2 + 1 < \alpha_2 + 3/2 = H_2.$$

If the summation S'_T does not include a sum over index 1, that is, if $1 \notin T$, one has

$$\begin{aligned} |(S'_T a_\pi)(\mathbf{i})| &\leq C \sum_{\mathbf{u} > \mathbf{0}} (i_1 + \dots + i_{m-l} + u_1 + \dots + u_l)^{\alpha_1} i_1^{\alpha_2} \\ &\leq C(i_1 + \dots + i_{m-l})^{\alpha_1 + l} i_1^{\alpha_2}, \end{aligned}$$

by (44), which also leads to an $H^* < \alpha_2 + 3/2 = H_2$. □

Proof of Case 4. Same as Case 3. □

Proof of Case 5. We consider first in Part 1 all cases of $S'_T a_\pi$ in (53) which contribute to the limit, and in Part 2 negligible cases.

Part 1 of Case 5: Suppose that π can be split into π_1 and π_2 which satisfy the following: the sub-partition $\pi_1 = \{P_1, \dots, P_{m_1}\}$ is a partition of $\{1, \dots, k_1\}$, such that each P_j satisfies $|P_j| \leq 2$, and at least one $|P_j| = 1$, $j = 1, \dots, m_1$.

Thus suppose without loss of generality that $|P_1| = 2, \dots, |P_r| = 2$, $0 \leq r < m_1$, and $|P_{r+1}| = \dots = |P_{m_1}| = 1$. Require that the sub-partition π_2 belongs to \mathcal{C}_2^2 , where \mathcal{C}_2^2 is the collection of partitions of $\{k_1 + 1, \dots, k_1 + \dots + k_2\}$ such that each set in π_2 contains at least 2 elements. \mathcal{C}_2^2 is nonempty because $k_2 \geq 2$. Let $T = \{1, \dots, r, m_1 + 1, \dots, m_1 + m_2\}$. Setting $\mathbf{i} = (i_1, \dots, i_{m_1-r})$, $\mathbf{u} = (u_1, \dots, u_r) \in \mathbb{Z}_+^r$ and $\mathbf{v} = (v_1, \dots, v_{m_2}) \in \mathbb{Z}_+^{m_2}$, one can write

$$\begin{aligned} (S'_T a_\pi)(\mathbf{i}) &= \sum_{\substack{u_p \neq u_q, u_p \neq i_q, u_p \neq v_q, \\ v_p \neq v_q, v_p \neq i_q, \mathbf{u}, \mathbf{v} > \mathbf{0}}} a^{(1)}(u_1, u_1, \dots, u_r, u_r, i_1, \dots, i_{m_1-r}) a_{\pi_2}^{(2)}(\mathbf{v}) \\ &= \sum_{u_p \neq u_q, u_p \neq v_q, v_p \neq v_q, \mathbf{u}, \mathbf{v} > \mathbf{0}} a^{(1)}(u_1, u_1, \dots, u_r, u_r, i_1, \dots, i_{m_1-r}) a_{\pi_2}^{(2)}(\mathbf{v}) - R_1(\mathbf{i}) \\ &= \sum_{u_p \neq u_q, \mathbf{u} > \mathbf{0}} a^{(1)}(u_1, u_1, \dots, u_r, u_r, i_1, \dots, i_{m_1-r}) \sum_{v_p \neq v_q, \mathbf{v} > \mathbf{0}} a_{\pi_2}^{(2)}(\mathbf{v}) - R_1(\mathbf{i}) - R_2(\mathbf{i}) \end{aligned} \quad (58)$$

for $i_p \neq i_q$. Relation (58) has three terms. We shall now apply Proposition 2.6 to the first term. Summing over all possible values of r , one gets a NCLT with Hurst index $H = \alpha_1 + k_1/2 + 1$, where the limit is $Z(t) := c_2 \sum_{0 \leq r < k_1/2} d_{k,r} Z_{k_1-2r}(t)$, where $Z_{k_1-2r}(t)$ is defined in (15) with $g_r = g_r^{(1)}$. Taking into account that in this setting, $c(\mathbf{p}, \mathbf{j})$ in (52) and (53) is

$$\binom{p_1}{0} \dots \binom{p_r}{0} \binom{p_{r+1}}{1} \dots \binom{p_{m_1}}{1} \binom{p_{m_1+1}}{0} \dots \binom{p_{m_1+m_2}}{0} (\mu_2)^r \mu_{p_{m_1+1}} \dots \mu_{p_{m_1+m_2}} =: \mu_{\pi_2},$$

since $\mu_2 = 1$, $p_1 = \dots = p_r = 2$ and $p_{r+1} = \dots = p_{m_1} = 1$, one gets the nonzero constant c_2 in (26). As in (16), we can express the limit $Z(t)$ as a centered Wiener-Stratonovich integral.

We shall now show that R_1 and R_2 in (58) lead only to terms with Hurst indices strictly less than $H = \alpha_1 + k_1/2 + 1$ in the sense of Definition 4.3, so they are negligible compared to the first term, and hence they do not contribute to the limit.

By identifying some of the u and v variables to the i variables in the first term of (58) and using the fact $a^{(j)}(\mathbf{i}) \leq C \|\mathbf{i}\|^{\alpha_j}$, one can see that one of the terms (a coefficient on $\mathbb{Z}_+^{m_1-r}$) in R_1 is bounded by:

$$\sum_{u_p \neq u_q, u_p \neq v_q, v_p \neq v_q, \mathbf{u}, \mathbf{v} > \mathbf{0}} C(\|\mathbf{u}\| + \|\mathbf{i}\|)^{\alpha_1} (\|\mathbf{v}\| + \|\mathbf{i}'\|)^{\alpha_2}, \quad (59)$$

where $\mathbf{u} = (u_1, \dots, u_{r-s_1})$, $\mathbf{i} = (i_1, \dots, i_{m_1-r})$, $\mathbf{v} = (v_1, \dots, v_{m_2-s_2})$, $\mathbf{i}' = (i_1, \dots, i_t)$, where $0 \leq s_1 \leq r \wedge (m_1 - r)$, $0 \leq t \leq s_2 \leq m_2 \wedge (m_1 - r)$. If $t = 0$, then $s_2 = 0$, and in addition, either $s_1 > 0$ or $s_2 > 0$. Note that \mathbf{i}' is a sub-vector of \mathbf{i} .

By (44), the term (59) is bounded by

$$\sum_{\mathbf{u}, \mathbf{v} > \mathbf{0}} C(\|\mathbf{u}\| + \|\mathbf{i}\|)^{\alpha_1} (\|\mathbf{v}\| + \|\mathbf{i}'\|)^{\alpha_2} \leq \begin{cases} C \|\mathbf{i}\|^{\alpha_1 + r - s_1} & \text{if } t = 0; \\ C \|\mathbf{i}\|^{\alpha_1 + r - s_1} \|\mathbf{i}'\|^{\alpha_2 + m_2 - s_2} & \text{if } t > 0. \end{cases}$$

When $t = s_2 = 0$, one must have $s_1 > 0$, and so the term yields an

$$H^* = \alpha_1 + r - s_1 + (m_1 - r)/2 + 1 = \alpha_1 + (r + m_1)/2 + 1 - s_1 < \alpha_1 + k_1/2 + 1,$$

because

$$r + m_1 = 2r + (m_1 - r) = k_1.$$

When $s_2 \geq t > 0$, it yields an

$$\begin{aligned} H^* &= \alpha_1 + r - s_1 + \alpha_2 + m_2 - s_2 + (m_1 - r)/2 + 1 = \alpha_1 + (m_1 + r)/2 + 1 + \alpha_2 + m_2 - s_1 - s_2 \\ &\leq \alpha_1 + k_1/2 + 1 + \alpha_2 + k_2/2 - s_1 - s_2 < \alpha_1 + k_1/2 + 1, \end{aligned}$$

since $2m_2 \leq k_2$ due to $\pi_2 \in \mathcal{C}_2^2$, and where the last inequality is due to the assumption $\alpha_2 < -k_2/2$.

We now examine R_2 in (58). By identifying some of the u variables to the v variables of the first term in (58), one term of R_2 can be bounded by

$$\sum_{u_p \neq u_q, v_p \neq v_q, \mathbf{u}, \mathbf{v} > \mathbf{0}} C(\|\mathbf{u}\| + \|\mathbf{v}_1\| + \|\mathbf{i}\|)^{\alpha_1} (\|\mathbf{v}_1\| + \|\mathbf{v}_2\|)^{\alpha_2},$$

where $\mathbf{u} = (u_1, \dots, u_{r-s})$, $\mathbf{v}_1 = (v_1, \dots, v_s)$, $\mathbf{v}_2 = (v_{s+1}, \dots, v_{m_2})$ and $\mathbf{i} = (i_1, \dots, i_{m_1-r})$, where $1 \leq s \leq (r \wedge m_2)$. By using (44), and then (47) and (45), this term is bounded by

$$\sum_{\mathbf{u} > \mathbf{0}, \mathbf{v}_1 > \mathbf{0}, \mathbf{v}_2 > \mathbf{0}} C(\|\mathbf{u}\| + \|\mathbf{v}_1\| + \|\mathbf{i}\|)^{\alpha_1} (\|\mathbf{v}_1\| + \|\mathbf{v}_2\|)^{\alpha_2} \leq \sum_{\mathbf{v}_1 > \mathbf{0}} C(\|\mathbf{v}_1\| + \|\mathbf{i}\|)^{\alpha_1 + r - s} \|\mathbf{v}_1\|^{\alpha_2 + m_2 - s} \leq C \|\mathbf{i}\|^{\alpha_1 + r - s + (s-1)/2},$$

which yields an

$$H^* = \alpha_1 + r - s/2 - 1/2 + (m_1 - r)/2 + 1 = \alpha_1 + (m_1 + r)/2 + 1 - s/2 - 1/2 < \alpha_1 + k_1/2 + 1.$$

So neither R_1 nor R_2 contributes in the limit.

Part 2 of Case 5. Suppose now that π and T which are *not* as in Part 1. To determine these cases, note that one can always bound $|(S'_T a_\pi)(\mathbf{i})|$ by

$$C \sum_{\mathbf{u} > \mathbf{0}} (\|\mathbf{i}_1\| + \|\mathbf{i}_2\| + \|\mathbf{u}_1\| + \|\mathbf{u}_2\|)^{\alpha_1} (\|\mathbf{i}_1\| + \|\mathbf{i}_3\| + \|\mathbf{u}_1\| + \|\mathbf{u}_3\|)^{\alpha_2} \quad (60)$$

where $\mathbf{i}_j \in \mathbb{Z}_+^{s_j}$, $\mathbf{u}_j \in \mathbb{Z}_+^{t_j}$, and where $s_1 + s_2 + s_3 > 0$ (at least one i variable must remain), and

$$s_1 + s_2 + t_1 + 2t_2 \leq k_1, \quad s_1 + s_3 + t_1 + 2t_3 \leq k_2.$$

Thus the variables in \mathbf{u}_2 are at least paired within $a^{(1)}$, and the variables in \mathbf{u}_3 are at least paired within $a^{(2)}$.

We note that in Part 1, we had $s_1 = s_3 = t_1 = 0$, and $s_t + 2t_2 = k_1$. Thus to avoid the situation considered in Part 1, we require:

$$\text{if } s_1 = s_3 = t_1 = 0, \text{ then } s_2 + 2t_2 < k_1. \quad (61)$$

As we have dealt with R_1 and R_2 before, by properly applying (44)-(47), the bound in (60) yields an $H^* < H_1 = \alpha_1/2 + k_1/2 + 1$. To check this, we consider the following exhaustive cases:

- (a) either $s_1 > 0$, or “ $s_1 = 0, s_2 > 0, s_3 > 0$ ”;
- (b) $s_1 = s_2 = 0, s_3 > 0$;
- (c) $s_1 = s_3 = 0, s_2 > 0$ but $s_2 + 2t_2 < k_1$;

Note that in case (c), if $s_2 + 2t_2 = k_1$ then $t_1 = 0$, which would contradict (61).

In case (a), for example if $s_1 > 0$, by applying (44) on the sum over \mathbf{u}_2 and \mathbf{u}_3 , and then (47) on the sum over \mathbf{u}_1 , we can bound (60) by

$$C(\|\mathbf{i}_1\| + \|\mathbf{i}_2\|)^{\alpha_1+t_1/2+t_2}(\|\mathbf{i}_1\| + \|\mathbf{i}_3\|)^{\alpha_2+t_1/2+t_3}$$

This yields an

$$\begin{aligned} H^* &= \alpha_1 + \alpha_2 + t_1 + t_2 + t_3 + (s_1 + s_2 + s_3)/2 + 1 \\ &= \alpha_1 + (s_1 + s_2 + t_1 + 2t_2)/2 + 1 + \alpha_2 + (s_3 + t_1 + 2t_3)/2 \\ &\leq \alpha_1 + k_1/2 + 1 + \alpha_2 + k_2/2 < \alpha_1 + k_1/2 + 1. \end{aligned} \tag{62}$$

In case (b), (60) becomes $C \sum_{\mathbf{u} > \mathbf{0}} (\|\mathbf{u}_1\| + \|\mathbf{u}_2\|)^{\alpha_1} (\|\mathbf{i}_3\| + \|\mathbf{u}_1\| + \|\mathbf{u}_3\|)^{\alpha_2}$ which we can bound by

$$C \sum_{\mathbf{u}_1 > \mathbf{0}} \|\mathbf{u}_1\|^{\alpha_1+t_2} (\|\mathbf{i}_3\| + \|\mathbf{u}_1\|)^{\alpha_2+t_3} \leq \begin{cases} \|\mathbf{i}_3\|^{\alpha_2+(t_1-1)/2+t_3} & \text{if } \alpha_1 + t_1/2 + t_2 < -1/2; \\ \|\mathbf{i}_3\|^{\alpha_1+\alpha_2+t_1+t_2+t_3} & \text{if } -1/2 < \alpha_1 + t_1/2 + t_2 < 0. \end{cases}$$

where we need to apply first (44), then apply (47) if $t_1 \geq 2$, and finally apply either (45) for the first case or (46) for the second. Note that $\alpha_1 + t_1/2 + t_2 > -1/2$ only if $t_1/2 + t_2 = k_1/2$ since $-k_1/2 - 1/2 < \alpha_1 < -k_1/2$ and $t_1 + 2t_2 \leq k_1$. So this yields either an

$$\begin{aligned} H^* &= \alpha_2 + (t_1 - 1)_+/2 + t_3 + s_3/2 + 1 = \alpha_2 + (s_3 + t_1 + 2t_3)/2 + 1 + (t_1 - 1)_+/2 - t_1/2 \\ &\leq \alpha_2 + k_2/2 + 1 = H_2 < H_1 \end{aligned} \tag{63}$$

or an H^* as in (62).

Similarly in case (c), (60) is $\sum_{\mathbf{u} > \mathbf{0}} C(\|\mathbf{i}_2\| + \|\mathbf{u}_1\| + \|\mathbf{u}_2\|)^{\alpha_1} (\|\mathbf{u}_1\| + \|\mathbf{u}_3\|)^{\alpha_2}$, which can be bounded by

$$C \sum_{\mathbf{u}_1 > \mathbf{0}} (\|\mathbf{i}_2\| + \|\mathbf{u}_1\|)^{\alpha_1+t_2} \|\mathbf{u}_1\|^{\alpha_2+t_3} \leq \begin{cases} \|\mathbf{i}_2\|^{\alpha_1+(t_1-1)/2+t_2} & \text{if } \alpha_2 + t_1/2 + t_3 < -1/2; \\ \|\mathbf{i}_2\|^{\alpha_1+\alpha_2+t_1+t_2+t_3} & \text{if } -1/2 < \alpha_1 + t_1/2 + t_2 < 0. \end{cases}$$

So it yields either an

$$\begin{aligned} H^* &= \alpha_1 + (t_1 - 1)_+/2 + t_2 + s_2/2 + 1 = \alpha_1 + (s_2 + t_1 + 2t_2)/2 + 1 + (t_1 - 1)_+/2 - t_1/2 \\ &< \alpha_1 + k_1/2 + 1 = H_1, \end{aligned} \tag{64}$$

or an H^* as in (62). To get the strict inequality in (64), we use (61) when $t_1 = 0$, and use $(t_1 - 1)_+/2 < t_1/2$ when $t_1 > 0$. \square

Proof of Case 6. Same as Case 5. \square

Proof of Case 7. Since $H_1 = H_2$, both factors $a^{(1)}$ and $a^{(2)}$ may contribute to the limit. The proof is similar to Case 5, while the other term in the limit arises by exchanging of the role of $a^{(1)}$ and $a^{(2)}$ in the proof of Case 5. Note that because $H_1 = H_2$, the equality in “ \leq ” in (63) is attained whenever $t_1 = 0$ and $s_3 + 2t_3 = k_2$, a case which would then be included in the NCLT part of the proof. \square

Acknowledgments. This work was partially supported by the NSF grants DMS-1007616 and DMS-1309009 at Boston University.

References

- [1] S. Bai and M.S. Taqqu. Generalized Hermite processes, discrete chaos and limit theorems. *Stochastic Processes and Their Applications*, 124(4):1710–1739, 2014.
- [2] S. Bai and M.S. Taqqu. Convergence of long-memory discrete k-th order Volterra processes. *Preprint arXiv:1403.1903*, 2014.

- [3] J. Beran, Y. Feng, S. Ghosh, and R. Kulik. *Long-Memory Processes*. Springer, 2013.
- [4] R.L. Dobrushin and P. Major. Non-central limit theorems for non-linear functional of Gaussian fields. *Probability Theory and Related Fields*, 50(1):27–52, 1979.
- [5] P. Embrechts and M. Maejima. *Selfsimilar Processes*. Princeton University Press, 2002.
- [6] L. Giraitis, H.L. Koul, and D. Surgailis. *Large Sample Inference for Long Memory Processes*. World Scientific Publishing Company Incorporated, 2012.
- [7] Y. Hu and P. Meyer. Sur les intégrales multiples de Stratonovitch. *Séminaire de Probabilités, XXII, Lecture Notes in Math*, 1321:72–81, 1988.
- [8] M. Maejima and C.A. Tudor. Selfsimilar processes with stationary increments in the second Wiener chaos. *Probability and Mathematical Statistics*, 32(1):167–186, 2012.
- [9] D. Surgailis. Zones of attraction of self-similar multiple integrals. *Lithuanian Mathematical Journal*, 22(3):327–340, 1982.
- [10] M.S. Taqqu. Convergence of integrated processes of arbitrary Hermite rank. *Probability Theory and Related Fields*, 50(1):53–83, 1979.

Shuyang Bai *bsy9142@bu.edu*
Murad S. Taqqu *murad@bu.edu*
Department of Mathematics and Statistics
111 Cumminston Street
Boston, MA, 02215, US